The Mean Euler Characteristic and Excursion Probability of Gaussian Random Fields with Stationary Increments *

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November 29, 2012

Abstract

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and let $T \subset \mathbb{R}^N$ be a compact rectangle. Under $X(\cdot) \in C^2(\mathbb{R}^N)$ and certain additional regularity conditions, the mean Euler characteristic of the excursion set $A_u = \{t \in T : X(t) \geq u\}$, denoted by $\mathbb{E}\{\varphi(A_u)\}$, is derived. By applying the Rice method, it is shown that, as $u \to \infty$, the excursion probability $\mathbb{P}\{\sup_{t \in T} X(t) \geq u\}$ can be approximated by $\mathbb{E}\{\varphi(A_u)\}$ such that the error is exponentially smaller than $\mathbb{E}\{\varphi(A_u)\}$. This verifies the expected Euler characteristic heuristic for a large class of Gaussian random fields with stationary increments.

Key Words: Gaussian random fields, stationary increments, excursion probability, excursion set, Euler characteristic, super-exponentially small.

1 Introduction

Let $X = \{X(t), t \in T\}$ be a real-valued Gaussian random field on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where T is the parameter set. The study for excursion probability $\mathbb{P}\{\sup_{t \in T} X(t) \geq u\}$ is a classical but very important problem in probability theory and has many applications in statistics and related areas. Many authors have developed various methods for precise approximations of $\mathbb{P}\{\sup_{t \in T} X(t) \geq u\}$. These include the double sum method [Piterbarg (1996a)], the tube method [Sun (1993)], the Euler characteristic method [Adler (2000), Taylor and Adler (2003), Taylor et al. (2005), Adler and Taylor (2007)] and the Rice method [Azaïs and Delmas (2002), Azaïs and Wschebor (2005, 2008, 2009)].

For a unit-variance smooth Gaussian random field $X = \{X(t), t \in T\}$ parameterized on manifold T, Adler and Taylor (2007, Theorem 14.3.3) proved, under certain conditions on the regularity of X and topology of T, the following approximation:

$$\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} = \mathbb{E}\{\varphi(A_u)\}\big(1 + o(e^{-\alpha u^2})\big), \quad \text{as } u \to \infty, \tag{1.1}$$

where $\varphi(A_u)$ is the Euler characteristic of excursion set $A_u = \{t \in T : X(t) \geq u\}$ and $\alpha > 0$ is a constant which relates to the curvature of the boundary of T and the second order partial derivatives of X. This verifies the "Expected Euler Characteristic Heuristic" for unit-variance smooth Gaussian random fields. See also Piterbarg (1996a, 1996b), Takemura and Kurki (2002), Taylor and Adler (2003), Taylor et al. (2005) for similar results in special cases.

The approximation (1.1) is remarkable and very accurate, since $\mathbb{E}\{\varphi(A_u)\}$ is computable and the error is exponentially smaller than this principal term. It has been applied for P-value approximation in many statistical applications to brain imaging, cosmology, and environmental sciences. We refer to Adler and Taylor (2007) and its forthcoming companion Adler, Taylor and Worsley (2012) for further information. However the above requirement of "constant variance" on the Gaussian random fields is too restrictive for many applications and excludes some important Gaussian random fields such as those with stationary increments [see Section 2 below], or more generally, Gaussian random intrinsic functions [Matheron (1973), Stein (1999, 2012)]. If the constant variance condition is not satisfied, then the formulas for computing $\mathbb{E}\{\varphi(A_u)\}$ [cf. Theorem 12.4.1 and Theorem 12.4.2 in Adler and Taylor (2007)] cannot be applied, and little had been known on whether the approximation (1.1) still holds. The only exception is Azaïs and Wschebor (2008, Theorem 5), where they proved (1.1) for a smooth Gaussian random field whose maximum variance is attained in the interior of T.

In this paper, let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered real-valued Gaussian random field with stationary increments and let $T \subset \mathbb{R}^N$ be a rectangle. Our objectives are to compute $\mathbb{E}\{\varphi(A_u)\}$ and to show that it can be applied to give an accurate approximation for the excursion probability $\mathbb{P}\{\sup_{t \in T} X(t) \geq u\}$ and, in particular, (1.1) holds for smooth Gaussian random fields with stationary increments.

The paper is organized as follows. Firstly, in Section 2, we provide some preliminaries on Gaussian random fields with stationary increments and prove some basic lemmas. These lemmas are derived from the spectral representation of the random fields and will be useful for proving the main results in Section 3 and Section 4.

In Section 3 we compute the mean Euler characteristic $\mathbb{E}\{\varphi(A_u)\}$ by applying the Kac-Rice metatheorem in Adler and Taylor (2007, Theorem 11.2.1) [see also Adler and Taylor (2011, Theorem 4.1.1)]. The computation of $\mathbb{E}\{\varphi(A_u)\}$ involves the conditional expectation of the determinant of $\nabla^2 X(t)$ given $X(t)$ and $\nabla X(t)$, which is more complicated for random

fields with non-constant variance function. For Gaussian random fields with stationary increments, we are able to make use of the properties of ∇X and $\nabla^2 X$ to provide an explicit formula in Theorem 3.2 for $\mathbb{E}\{\varphi(A_u)\}$, using only up to the second-order derivatives of the covariance function.

Section 4 is the core part of this paper. Theorems 4.7 and 4.8 provide approximations to the excursion probability which are analogous to (1.1) for Gaussian random fields with stationary increments. Since these random fields do not have constant variance, it is not clear if the original method for proving Theorem 14.3.3 in Adler and Taylor (2007) is still applicable. Instead, our argument is based on the Rice method in Azaïs and Delmas (2002) [see also Adler and Taylor (2007, p.96-99)]. More specifically, we decompose the rectangle T into several faces of lower dimensions and then apply the idea of Piterbarg (1996b) and the Bonferroni inequality to derive upper and lower bounds for $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ in terms of the the number of extended outward maxima [see (4.1), (4.2)] and the local maxima [see (4.3)], respectively. The main idea is to show that, in both cases, the upper bound makes the major contribution for estimating $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$, and the last two terms in the lower bounds in (4.2) and (4.3) are super-exponentially small. Under an additional condition on the variogram of X, we apply (4.3) to obtain in Theorem 4.7 an expansion for the excursion probability which is, in spirit, similar to the case of stationary Gaussian fields [cf. (14.0.3) in Adler and Taylor (2007)]. Theorem 4.8 establishes a general approximation to $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ in terms of $\mathbb{E}\{\varphi(A_u)\}$, which verifies the "Expected Euler Characteristic Heuristic" for smooth Gaussian random fields with stationary increments.

Section 5 provides further remarks on the main results and some examples where significant simplifications can be made. In Example 5.3 and Example 5.4, we show that if the variance function of the random field attains its maximum at a unique point, then one can apply the Laplace method to derive a first-order approximation for the excursion probability explicitly. Finally, the Appendix contains proofs of some auxiliary lemmas.

2 Gaussian Fields with Stationary Increments

2.1 Spectral Representation

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments. We assume that X has continuous covariance function $C(t, s) = \mathbb{E}\{X(t)X(s)\}$ and $X(0) = 0$. Then it is known [cf. Yaglom (1957)] that

$$C(t, s) = \int_{\mathbb{R}^N} (e^{i\langle t,\lambda\rangle} - 1)(e^{-i\langle s,\lambda\rangle} - 1)\, F(d\lambda) + \langle t, \Theta s\rangle \tag{2.1}$$

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where $\langle x, y \rangle$ is the ordinary inner product in $\mathbb{R}^N$, Θ is an $N \times N$ non-negative definite matrix and F is a non-negative symmetric measure on $\mathbb{R}^N \backslash \{0\}$ which satisfies

$$\int_{\mathbb{R}^N} \frac{\|\lambda\|^2}{1 + \|\lambda\|^2} F(d\lambda) < \infty. \tag{2.2}$$

Similarly to stationary random fields, the measure F and its density (if it exists) $f(\lambda)$ are called the *spectral measure* and *spectral density* of X, respectively.

By (2.1) we see that X has the following stochastic integral representation

$$X(t) = \int_{\mathbb{R}^N} (e^{i\langle t, \lambda \rangle} - 1) W(d\lambda) + \langle \mathbf{Y}, t \rangle, \tag{2.3}$$

where $\mathbf{Y}$ is an N-dimensional Gaussian random vector and W is a complex-valued Gaussian random measure (independent of $\mathbf{Y}$) with F as its control measure. It is known that many probabilistic, analytic and geometric properties of a Gaussian field with stationary increments can be described in terms of its spectral measure F and, on the other hand, various interesting Gaussian random fields can be constructed by choosing their spectral measures appropriately. See Xiao (2009), Xue and Xiao (2011) and the references therein for more information.

For simplicity we assume that $\mathbf{Y} = 0$. It follows from (2.1) that the *variogram* ν of X is given by

$$\nu(h) := \mathbb{E}(X(t + h) - X(t))^2 = 2 \int_{\mathbb{R}^N} (1 - \cos \langle h, \lambda \rangle) F(d\lambda). \tag{2.4}$$

Mean-square directional derivatives and sample path differentiability of Gaussian random fields have been well studied. See, for example, Adler (1981), Adler and Taylor (2007), Potthoff (2010), Xue and Xiao (2011). In particular, general sufficient conditions for a Gaussian random field to have a modification whose sample functions are in $C^k(\mathbb{R}^N)$ are given by Adler and Taylor (2007). For a Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ with stationary increments, Xue and Xiao (2011) provided conditions for its sample path differentiability in terms of the spectral density function $f(\lambda)$. Similar argument can be applied to give the following spectral condition for the sample functions of X to be in $C^k(\mathbb{R}^N)$, whose proof is given in Cheng (2013) and is omitted here.

Proposition 2.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments and let k_i $(1 \le i \le N)$ be non-negative integers. If there is a constant $\varepsilon > 0$ such that*

$$\int_{\{\|\lambda\| \ge 1\}} \prod_{i=1}^N |\lambda_i|^{2k_i + \varepsilon} F(d\lambda) < \infty, \tag{2.5}$$

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then X has a modification $\widetilde{X}$ such that the partial derivative $\frac{\partial^k \widetilde{X}(t)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N}}$ is continuous on $\mathbb{R}^N$ almost surely, where $k = \sum_{i=1}^N k_i$. Moreover, for any compact rectangle $T \subset \mathbb{R}^N$ and any $\varepsilon' \in (0, \varepsilon \wedge 1)$, there exists a constant c_1 such that

$$\mathbb{E}\left(\frac{\partial^k \widetilde{X}(t)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N}} - \frac{\partial^k \widetilde{X}(s)}{\partial s_1^{k_1} \cdots \partial s_N^{k_N}} \right)^2 \leq c_1 \|t - s\|^{\varepsilon'}, \quad \forall t, s \in T. \tag{2.6}$$

For simplicity we will not distinguish X from its modification $\widetilde{X}$. As a consequence of Proposition 2.1, we see that, if $X = \{X(t), t \in \mathbb{R}^N\}$ has a spectral density $f(\lambda)$ which satisfies

$$f(\lambda) = O\left(\frac{1}{\|\lambda\|^{N+2k+H}} \right) \quad \text{as } \|\lambda\| \to \infty, \tag{2.7}$$

for some integer $k \geq 1$ and $H \in (0,1)$, then the sample functions of X are in $C^k(\mathbb{R}^N)$ a.s. Further examples of anisotropic Gaussian random fields which may have different smoothness along different directions can be found in Xue and Xiao (2011).

When $X(\cdot) \in C^2(\mathbb{R}^N)$ almost surely, we write $\frac{\partial X(t)}{\partial t_i} = X_i(t)$ and $\frac{\partial^2 X(t)}{\partial t_i \partial t_j} = X_{ij}(t)$. Denote by $\nabla X(t)$ and $\nabla^2 X(t)$ the column vector $(X_1(t), \ldots, X_N(t))^T$ and the $N \times N$ matrix $(X_{ij}(t))_{i,j=1,\ldots,N}$, respectively. It follows from (2.1) that for every $t \in \mathbb{R}^N$,

$$\lambda_{ij} := \int_{\mathbb{R}^N} \lambda_i \lambda_j F(d\lambda) = \left. \frac{\partial^2 C(t,s)}{\partial t_i \partial s_j} \right|_{s=t} = \mathbb{E}\{X_i(t) X_j(t)\}. \tag{2.8}$$

Let $\Lambda = (\lambda_{ij})_{i,j=1,\ldots,N}$, then (2.8) shows that $\Lambda = \text{Cov}(\nabla X(t))$ for all t. In particular, the distribution of $\nabla X(t)$ is independent of t. Let

$$\lambda_{ij}(t) := \int_{\mathbb{R}^N} \lambda_i \lambda_j \cos \langle t, \lambda \rangle \, F(d\lambda), \qquad \Lambda(t) := (\lambda_{ij}(t))_{i,j=1,\ldots,N}.$$

Then we have

$$\lambda_{ij}(t) - \lambda_{ij} = \int_{\mathbb{R}^N} \lambda_i \lambda_j (\cos \langle t, \lambda \rangle - 1) \, F(d\lambda) = \left. \frac{\partial^2 C(t,s)}{\partial t_i \partial t_j} \right|_{s=t} = \mathbb{E}\{X(t) X_{ij}(t)\},$$

or equivalently, $\Lambda(t) - \Lambda = \mathbb{E}\{X(t) \nabla^2 X(t)\}$.

2.2 Hypotheses and Some Important Properties

Let $T = \prod_{i=1}^N [a_i, b_i]$ be a compact rectangle in $\mathbb{R}^N$, where $a_i < b_i$ for all $1 \leq i \leq N$ and $0 \notin T$ [the case of $0 \in T$ will be discussed in Remark 5.1]. In addition to the stationary increments, we will make use of the following conditions on X:

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(**H**1). $X(\cdot) \in C^2(T)$ almost surely and its second derivatives satisfy the *uniform mean-square Hölder condition*: there exist constants L and $\eta \in (0,1]$ such that

$$\mathbb{E}(X_{ij}(t) - X_{ij}(s))^2 \leq L\|t-s\|^{2\eta}, \quad \forall t,s \in T, \; i,j = 1,\ldots,N. \tag{2.9}$$

(**H**2). For every $t \in T$, the matrix $\Lambda - \Lambda(t)$ is non-degenerate.

(**H**3). For every pair $(t,s) \in T^2$ with $t \neq s$, the Gaussian random vector

$$(X(t), \nabla X(t), X_{ij}(t), \; X(s), \nabla X(s), X_{ij}(s), 1 \leq i \leq j \leq N)$$

is non-degenerate.

(**H**3′). For every $t \in T$, $(X(t), \nabla X(t), X_{ij}(t), 1 \leq i \leq j \leq N)$ is non-degenerate.

Clearly, by Proposition 2.1, condition (**H**1) is satisfied if (2.7) holds for $k=2$. Also note that (**H**3) implies (**H**3′). We shall use conditions (**H**1), (**H**2) and (**H**3) to prove Theorems 4.7 and 4.8. Condition (**H**3′) will be used for computing $\mathbb{E}\{\varphi(A_u)\}$ in Theorem 3.2.

The following lemma shows that for Gaussian fields with stationary increments, (**H**2) is equivalent to $\Lambda - \Lambda(t)$ being positive definite.

Lemma 2.2 *For every $t \neq 0$, $\Lambda - \Lambda(t)$ is non-negative definite. Hence, under (**H**2), $\Lambda - \Lambda(t)$ is positive definite.*

Proof　Let $t \neq 0$ be fixed. For any $(a_1,\ldots,a_N) \in \mathbb{R}^N \backslash \{0\}$,

$$\sum_{i,j=1}^N a_i a_j (\lambda_{ij} - \lambda_{ij}(t)) = \int_{\mathbb{R}^N} \left(\sum_{i=1}^N a_i \lambda_i\right)^2 (1 - \cos\langle t,\lambda\rangle) \, F(\lambda). \tag{2.10}$$

Since $(\sum_{i=1}^N a_i \lambda_i)^2 (1 - \cos\langle t,\lambda\rangle) \geq 0$ for all $\lambda \in \mathbb{R}^N$, (2.10) is always non-negative, which implies $\Lambda - \Lambda(t)$ is non-negative definite. If (**H**2) is satisfied, then all the eigenvalues of $\Lambda - \Lambda(t)$ are positive. This completes the proof. □

It follows from (2.10) that, if the spectral measure F is carried by a set of positive Lebesgue measure [i.e., there is a set $B \subset \mathbb{R}^N$ with positive Lebesgue measure such that such that $F(B) > 0$], then (**H**2) holds. Hence, (**H**2) is in fact a very mild condition for smooth Gaussian fields with stationary increments.

Lemma 2.2 and the following two lemmas indicate some significant properties of Gaussian fields with stationary increments. They will play important roles in later sections.

Lemma 2.3 *Let $t \in \mathbb{R}^N$ be fixed. Then for all i,j,k, the random variables $X_i(t)$ and $X_{jk}(t)$ are independent. Moreover, $\mathbb{E}\{X_{ij}(t)X_{kl}(t)\}$ is symmetric in i, j, k, l.*

Proof By (2.1), one can verify that for $t, s \in \mathbb{R}^N$,

$$\mathbb{E}\{X_i(t)X_{jk}(s)\} = \frac{\partial^3 C(t,s)}{\partial t_i \partial s_j \partial s_k} = \int_{\mathbb{R}^N} \lambda_i \lambda_j \lambda_k \sin\langle t - s, \lambda \rangle\, F(d\lambda).$$

Letting $s = t$ we see that $X_i(t)$ and $X_{jk}(t)$ are independent. Similarly, we have

$$\mathbb{E}\{X_{ij}(t)X_{kl}(s)\} = \frac{\partial^4 C(t,s)}{\partial t_i \partial t_j \partial s_k \partial s_l} = \int_{\mathbb{R}^N} \lambda_i \lambda_j \lambda_k \lambda_l \cos\langle t - s, \lambda \rangle\; F(d\lambda).$$

This implies the second conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following lemma is a directly consequence of Lemma 2.3.

Lemma 2.4 *Let $A = (a_{ij})_{1 \le i,j \le N}$ be a symmetric matrix, then*

$$\mathcal{S}_t(i,j,k,l) = \mathbb{E}\{(A\nabla^2 X(t)A)_{ij}(A\nabla^2 X(t)A)_{kl}\}$$

is a symmetric function of i, j, k, l.

3 The Mean Euler Characteristic

3.1 Related Existing Results and Notation

The rectangle $T = \prod_{i=1}^N [a_i, b_i]$ can be decomposed into several faces of lower dimensions. We use the same notation as in Adler and Taylor (2007, p.134). A face J of dimension k, is defined by fixing a subset $\sigma(J) \subset \{1, \dots, N\}$ of size k and a subset $\varepsilon(J) = \{\varepsilon_j, j \notin \sigma(J)\} \subset \{0,1\}^{N-k}$ of size $N - k$, so that

$$J = \{t = (t_1, \dots, t_N) \in T : a_j < t_j < b_j \text{ if } j \in \sigma(J),$$
$$t_j = (1 - \varepsilon_j)a_j + \varepsilon_j b_j \text{ if } j \notin \sigma(J)\}.$$

Denote by $\partial_k T$ the collection of all k-dimensional faces in T, then the interior of T is given by $\overset{\circ}{T} = \partial_N T$ and the boundary of T is given by $\partial T = \cup_{k=0}^{N-1} \cup_{J \in \partial_k T} J$. For $J \in \partial_k T$, denote by $\nabla X_{|J}(t)$ and $\nabla^2 X_{|J}(t)$ the column vector $(X_{i_1}(t), \dots, X_{i_k}(t))^T_{i_1, \dots, i_k \in \sigma(J)}$ and the $k \times k$ matrix $(X_{mn}(t))_{m,n \in \sigma(J)}$, respectively.

If $X(\cdot) \in C^2(\mathbb{R}^N)$ and it is a Morse function a.s. [cf. Definition 9.3.1 in Adler and Taylor (2007)], then according to Corollary 9.3.5 or page 211-212 in Adler and Taylor (2007), the Euler characteristic of the excursion set $A_u = \{t \in T : X(t) \ge u\}$ is given by

$$\varphi(A_u) = \sum_{k=0}^N \sum_{J \in \partial_k T} (-1)^k \sum_{i=0}^k (-1)^i \mu_i(J) \tag{3.1}$$

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with

$$\mu_i(J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \text{index}(\nabla^2 X_{|J}(t)) = i,$$
$$\varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\}, \tag{3.2}$$

where $\varepsilon_j^* = 2\varepsilon_j - 1$ and the index of a matrix is defined as the number of its negative eigenvalues. We also define

$$\widetilde{\mu}_i(J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \text{index}(\nabla^2 X_{|J}(t)) = i\}. \tag{3.3}$$

Let $\sigma_t^2 = \text{Var}(X(t))$ and let $\sigma_T^2 = \sup_{t \in T} \sigma_t^2$ be the maximum variance. For Gaussian fields with stationary increments, it follows from (2.4) that $\nu(t) = \sigma_t^2$. For $t \in J \in \partial_k T$, where $k \geq 1$, let

$$\Lambda_J = (\lambda_{ij})_{i,j \in \sigma(J)}, \quad \Lambda_J(t) = (\lambda_{ij}(t))_{i,j \in \sigma(J)},$$
$$\theta_t^2 = \text{Var}(X(t)|\nabla X_{|J}(t)), \quad \gamma_t^2 = \text{Var}(X(t)|\nabla X(t)), \tag{3.4}$$
$$\{J_1, \ldots, J_{N-k}\} = \{1, \ldots, N\}\backslash\sigma(J),$$
$$E(J) = \{(t_{J_1}, \ldots, t_{J_{N-k}}) \in \mathbb{R}^{N-k} : t_j \varepsilon_j^* \geq 0, j = J_1, \ldots, J_{N-k}\}.$$

Then for all $t \in J$,

$$\Lambda_J = \text{Cov}(\nabla X_{|J}(t)), \quad \Lambda_J(t) - \Lambda_J = \mathbb{E}\{X(t)\nabla^2 X_{|J}(t)\}. \tag{3.5}$$

Note that $\theta_t^2 \geq \gamma_t^2$ for all $t \in T$ and $\theta_t^2 = \gamma_t^2$ if $t \in \partial_N T$. If $\{t\} \in \partial_0 T$, then $\nabla X_{|J}(t)$ is not defined, in this case we set θ_t^2 as σ_t^2 by convention. Let $C_j(t)$ be the $(1, j+1)$ entry of $(\text{Cov}(X(t), \nabla X(t)))^{-1}$, i.e. $C_j(t) = M_{1,j+1}/\text{detCov}(X(t), \nabla X(t))$, where $M_{1,j+1}$ is the cofactor of the $(1, j+1)$ entry, $\mathbb{E}\{X(t)X_j(t)\}$, in the covariance matrix $\text{Cov}(X(t), \nabla X(t))$.

Denote by $H_k(x)$ the Hermite polynomial of order k, i.e., $H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k}(e^{-x^2/2})$. Then the following identity holds [cf. Adler and Taylor (2007, p.289)]:

$$\int_u^\infty H_k(x)e^{-x^2/2}\,dx = H_{k-1}(u)e^{-u^2/2}, \tag{3.6}$$

where $u > 0$ and $k \geq 1$. For a matrix A, we will use $|A|$ or $\det A$ to denote its determinant. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$ and $\Psi(u) = (2\pi)^{-1/2}\int_u^\infty e^{-x^2/2}dx$.

3.2 Computing the Mean Euler Characteristic

The following lemma is an analogue of Lemma 11.7.1 in Adler and Taylor (2007). It provides a key step for computing the mean Euler characteristic in Theorem 3.2, meanwhile, it has close connection with Theorem 4.7.

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Lemma 3.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments satisfying* (**H1**), (**H2**) *and* (**H3$'$**)*. Then for each $J \in \partial_k T$ with $k \geq 1$,*

$$\mathbb{E}\left\{\sum_{i=0}^{k}(-1)^i \widetilde{\mu}_i(J)\right\} = \frac{(-1)^k}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}} \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\left(\frac{u}{\theta_t}\right) e^{-u^2/(2\theta_t^2)} dt. \qquad (3.7)$$

Proof   Let $\mathcal{D}_i$ be the collection of all $k \times k$ matrices with index i. Recall the definition of $\widetilde{\mu}_i(J)$ in (3.3), thanks to (**H1**) and (**H3$'$**), we can apply the Kac-Rice metatheorem [cf. Theorem 11.2.1 or Corollary 11.2.2 in Adler and Taylor (2007)] to get that the left hand side of (3.7) becomes

$$\int_J p_{\nabla X_{|J}(t)}(0)dt \sum_{i=0}^{k}(-1)^i \mathbb{E}\{|\det\nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_i\}} \mathbb{1}_{\{X(t) \geq u\}} |\nabla X_{|J}(t) = 0\}. \qquad (3.8)$$

Note that on the event $\mathcal{D}_i$, the matrix $\nabla^2 X_{|J}(t)$ has i negative eigenvalues, which implies $(-1)^i|\det\nabla^2 X_{|J}(t)| = \det\nabla^2 X_{|J}(t)$. Also, $\cup_{i=0}^k\{\nabla^2 X_{|J}(t) \in \mathcal{D}_i\} = \Omega$ a.s., hence (3.8) equals

$$\int_J p_{\nabla X_{|J}(t)}(0)dt\, \mathbb{E}\{\det\nabla^2 X_{|J}(t) \mathbb{1}_{\{X(t) \geq u\}} |\nabla X_{|J}(t) = 0\}$$
$$= \int_J \frac{e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t} dt \int_u^\infty dx\, \mathbb{E}\{\det\nabla^2 X_{|J}(t)|X(t) = x, \nabla X_{|J}(t) = 0\}. \qquad (3.9)$$

Now we turn to computing $\mathbb{E}\{\det\nabla^2 X_{|J}(t)|X(t) = x, \nabla X_{|J}(t) = 0\}$. By Lemma 2.2, under (**H2**), $\Lambda - \Lambda(t)$ and hence $\Lambda_J - \Lambda_J(t)$ are positive definite for every $t \in J$. Thus there exists a $k \times k$ positive definite matrix Q_t such that

$$Q_t(\Lambda_J - \Lambda_J(t))Q_t = I_k, \qquad (3.10)$$

where I_k is the $k \times k$ identity matrix. By (3.5),

$$\mathbb{E}\{X(t)(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}\} = -(Q_t(\Lambda_J - \Lambda_J(t))Q_t)_{ij} = -\delta_{ij},$$

where δ_{ij} is the Kronecker delta function. One can write

$$\mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)|X(t) = x, \nabla X_{|J}(t) = 0\} = \mathbb{E}\{\det\Delta(t,x)\}, \qquad (3.11)$$

where $\Delta(t,x) = (\Delta_{ij}(t,x))_{i,j \in \sigma(J)}$ with all elements $\Delta_{ij}(t,x)$ being Gaussian variables. To study $\Delta(t,x)$, we only need to find its mean and covariance. Note that $\nabla X(t)$ and $\nabla^2 X(t)$ are independent by Lemma 2.3, then we apply Lemma 6.1 to obtain

$$\mathbb{E}\{\Delta_{ij}(t,x)\} = \mathbb{E}\{(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}|X(t) = x, \nabla X_{|J}(t) = 0\}$$
$$= (\mathbb{E}\{X(t)(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}\}, 0, \ldots, 0)(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(x, 0, \ldots, 0)^T \qquad (3.12)$$
$$= (-\delta_{ij}, 0, \ldots, 0)(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(x, 0, \ldots, 0)^T = -\frac{x}{\theta_t^2}\delta_{ij},$$

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where the last equality comes from the fact that the $(1,1)$ entry of $(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}$ is $\mathrm{det}\mathrm{Cov}(\nabla X_{|J}(t))/\mathrm{det}\mathrm{Cov}(X(t), \nabla X_{|J}(t)) = 1/\theta_t^2$. For the covariance, applying Lemma 6.1 again gives

$$
\begin{aligned}
&\mathbb{E}\{(\Delta_{ij}(t,x) - \mathbb{E}\{\Delta_{ij}(t,x)\})(\Delta_{kl}(t,x) - \mathbb{E}\{\Delta_{kl}(t,x)\})\} \\
&\quad = \mathbb{E}\{(Q_t \nabla^2 X_{|J}(t) Q_t)_{ij}(Q_t \nabla^2 X_{|J}(t) Q_t)_{kl}\} - (\mathbb{E}\{X(t)(Q_t \nabla^2 X_{|J}(t) Q_t)_{ij}\}, 0, \ldots, 0) \\
&\qquad \cdot (\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(\mathbb{E}\{X(t)(Q_t \nabla^2 X_{|J}(t) Q_t)_{kl}\}, 0, \ldots, 0)^T \\
&\quad = \mathcal{S}_t(i,j,k,l) - (-\delta_{ij}, 0, \ldots, 0)(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(-\delta_{kl}, 0, \ldots, 0)^T \\
&\quad = \mathcal{S}_t(i,j,k,l) - \frac{\delta_{ij}\delta_{kl}}{\theta_t^2},
\end{aligned}
$$

where $\mathcal{S}_t$ is a symmetric function of i, j, k, l by applying Lemma 2.4 with A replaced by Q_t. Therefore (3.11) becomes

$$
\mathbb{E}\left\{\frac{1}{\theta_t^k}\mathrm{det}(\theta_t Q_t(\nabla^2 X_{|J}(t))Q_t)\Big| X(t) = x, \nabla X_{|J}(t) = 0\right\} = \frac{1}{\theta_t^k}\mathbb{E}\left\{\mathrm{det}\left(\widetilde{\Delta}(t) - \frac{x}{\theta_t}I_k\right)\right\},
$$

where $\widetilde{\Delta}(t) = (\widetilde{\Delta}_{ij}(t))_{i,j\in\sigma(J)}$ and all $\widetilde{\Delta}_{ij}(t)$ are Gaussian variables satisfying

$$
\mathbb{E}\{\widetilde{\Delta}_{ij}(t)\} = 0, \quad \mathbb{E}\{\widetilde{\Delta}_{ij}(t)\widetilde{\Delta}_{kl}(t)\} = \theta_t^2 \mathcal{S}_t(i,j,k,l) - \delta_{ij}\delta_{kl}.
$$

By Corollary 11.6.3 in Adler and Taylor (2007), (3.11) is equal to $(-1)^k \theta_t^{-k} H_k(x/\theta_t)$, hence

$$
\begin{aligned}
&\mathbb{E}\{\mathrm{det}\nabla^2 X_{|J}(t)|X(t) = x, \nabla X_{|J}(t) = 0\} \\
&\quad = \mathbb{E}\{\mathrm{det}(Q_t^{-1} Q_t \nabla^2 X_{|J}(t) Q_t Q_t^{-1})|X(t) = x, \nabla X_{|J}(t) = 0\} \\
&\quad = |\Lambda_J - \Lambda_J(t)|\mathbb{E}\{\mathrm{det}(Q_t \nabla^2 X_{|J}(t) Q_t)|X(t) = x, \nabla X_{|J}(t) = 0\} \\
&\quad = \frac{(-1)^k}{\theta_t^k}|\Lambda_J - \Lambda_J(t)|H_k\left(\frac{x}{\theta_t}\right).
\end{aligned}
$$

Plugging this into (3.9) and applying (3.6), we obtain the desired result. $\qquad\square$

Theorem 3.2 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments such that* (**H**1), (**H**2) *and* (**H**3′) *are fulfilled. Then*

$$
\begin{aligned}
\mathbb{E}\{\varphi(A_u)\} &= \sum_{\{t\}\in\partial_0 T}\mathbb{P}(X(t) \geq u, \nabla X(t) \in E(\{t\})) + \sum_{k=1}^{N}\sum_{J\in\partial_k T}\frac{1}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} \\
&\quad \times \int_J dt \int_u^\infty dx \int\cdots\int_{E(J)} dy_{J_1}\cdots dy_{J_{N-k}}\frac{|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k} \\
&\quad \times H_k\left(\frac{x}{\gamma_t} + \gamma_t C_{J_1}(t)y_{J_1} + \cdots + \gamma_t C_{J_{N-k}}(t)y_{J_{N-k}}\right) \\
&\quad \times p_{X(t), X_{J_1}(t), \ldots, X_{J_{N-k}}(t)}(x, y_{J_1}, \ldots, y_{J_{N-k}}|\nabla X_{|J}(t) = 0).
\end{aligned}
\tag{3.13}
$$

Proof   According to Corollary 11.3.2 in Adler and Taylor (2007), (**H**1) and (**H3′**) imply that X is a Morse function a.s. It follows from (3.1) that

$$\mathbb{E}\{\varphi(A_u)\} = \sum_{k=0}^{N} \sum_{J \in \partial_k T} (-1)^k \mathbb{E}\left\{\sum_{i=0}^{k} (-1)^i \mu_i(J)\right\}. \tag{3.14}$$

If $J \in \partial_0 T$, say $J = \{t\}$, it turns out $\mathbb{E}\{\mu_0(J)\} = \mathbb{P}(X(t) \geq u, \nabla X(t) \in E(\{t\}))$. If $J \in \partial_k T$ with $k \geq 1$, we apply the Kac-Rice metatheorem to obtain that the expectation on the right hand side of (3.14) becomes

$$\int_J p_{\nabla X_{|J}(t)}(0)dt \sum_{i=0}^{k}(-1)^i \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_i\}} \mathbb{1}_{\{(X_{J_1}(t),\ldots,X_{J_{N-k}}(t)) \in E(J)\}}$$

$$\times \mathbb{1}_{\{X(t) \geq u\}} |\nabla X_{|J}(t) = 0\}$$

$$= \frac{1}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} \int_J dt \int_u^\infty dx \int \cdots \int_{E(J)} dy_{J_1} \cdots dy_{J_{N-k}} \tag{3.15}$$

$$\times \mathbb{E}\{\det \nabla^2 X_{|J}(t)|X(t) = x, X_{J_1}(t) = y_{J_1}, \ldots, X_{J_{N-k}}(t) = y_{J_{N-k}}, \nabla X_{|J}(t) = 0\}$$

$$\times p_{X(t),X_{J_1}(t),\ldots,X_{J_{N-k}}(t)}(x, y_{J_1}, \ldots, y_{J_{N-k}}|\nabla X_{|J}(t) = 0).$$

For fixed t, let Q_t be the positive definite matrix in (3.10). Then, similarly to the proof in Lemma 3.1, we can write

$$\mathbb{E}\{\det(Q_t \nabla^2 X_{|J}(t)Q_t)|X(t) = x, X_{J_1}(t) = y_{J_1}, \ldots, X_{J_{N-k}} = y_{J_{N-k}}, \nabla X_{|J}(t) = 0\}$$

as $\mathbb{E}\{\det \overline{\Delta}(t, x)\}$, where $\overline{\Delta}(t, x)$ is a matrix consisting of Gaussian entries $\overline{\Delta}_{ij}(t, x)$ with mean

$$\mathbb{E}\{(Q_t \nabla^2 X_{|J}(t)Q_t)_{ij}|X(t) = x, X_{J_1}(t) = y_{J_1}, \ldots, X_{J_{N-k}} = y_{J_{N-k}}, \nabla X_{|J}(t) = 0\}$$

$$= (-\delta_{ij}, 0, \ldots, 0)(\text{Cov}(X(t), X_{J_1}(t), \ldots, X_{J_{N-k}}(t), \nabla X_{|J}(t)))^{-1}$$

$$\cdot (x, y_{J_1}, \ldots, y_{J_{N-k}}, 0, \ldots, 0)^T \tag{3.16}$$

$$= -\frac{\delta_{ij}}{\gamma_t^2}(x + \gamma_t^2 C_{J_1}(t)y_{J_1} + \cdots + \gamma_t^2 C_{J_{N-k}}(t)y_{J_{N-k}}),$$

and covariance

$$\mathbb{E}\{(\overline{\Delta}_{ij}(t, x) - \mathbb{E}\{\overline{\Delta}_{ij}(t, x)\})(\overline{\Delta}_{kl}(t, x) - \mathbb{E}\{\overline{\Delta}_{kl}(t, x)\})\} = \mathcal{S}_t(i, j, k, l) - \frac{\delta_{ij}\delta_{kl}}{\gamma_t^2}.$$

Following the same procedure in the proof of Lemma 3.1, we obtain that the last conditional expectation in (3.15) is equal to

$$\frac{(-1)^k|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k} H_k\left(\frac{x}{\gamma_t} + \gamma_t C_{J_1}(t)y_{J_1} + \cdots + \gamma_t C_{J_{N-k}}(t)y_{J_{N-k}}\right). \tag{3.17}$$

Plug this into (3.15) and (3.14), yielding the desired result. □

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Remark 3.3 Usually, for nonstationary (including constant-variance) Gaussian field X on $\mathbb{R}^N$, its mean Euler characteristic involves at least the third-order derivatives of the covariance function. For Gaussian random fields with stationary increments, as shown in Lemma 2.3, $\mathbb{E}\{X_{ij}(t)X_k(t)\} = 0$ and $\mathbb{E}\{X_{ij}(t)X_{kl}(t)\}$ is symmetric in i, j, k, l, so the mean Euler characteristic becomes relatively simpler, contains only up to the second-order derivatives of the covariance function. In various practical applications, (3.13) could be simplified with only an exponentially smaller difference, see the discussions in Section 5.

4 Excursion Probability

4.1 Preliminaries

As in Section 3.1, we decompose T into several faces as $T = \bigcup_{k=0}^{N} \partial_k T = \bigcup_{k=0}^{N} \bigcup_{J \in \partial_k T} J$. For each $J \in \partial_k T$, define the number of *extended outward maxima* above level u as

$$M_u^E(J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \text{index}(\nabla^2 X_{|J}(t)) = k,$$
$$\varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\}.$$

In fact, $M_u^E(J)$ is the same as $\mu_k(J)$ defined in (3.2) with $i = k$. We will make use of the following lemma, whose proof is given in the Appendix.

Lemma 4.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field satisfying* (**H**1) *and* (**H**3′), *then for any $u > 0$,*

$$\left\{ \sup_{t \in T} X(t) \geq u \right\} = \bigcup_{k=0}^{N} \bigcup_{J \in \partial_k T} \{M_u^E(J) \geq 1\} \text{ a.s.}$$

It follows from Lemma 4.1 that

$$\mathbb{P}\left\{ \sup_{t \in T} X(t) \geq u \right\} \leq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{P}\{M_u^E(J) \geq 1\} \leq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u^E(J)\}. \tag{4.1}$$

On the other hand, by the Bonferroni inequality,

$$\mathbb{P}\left\{ \sup_{t \in T} X(t) \geq u \right\} \geq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{P}\{M_u^E(J) \geq 1\} - \sum_{J \neq J'} \mathbb{P}\{M_u^E(J) \geq 1, M_u^E(J') \geq 1\}.$$

Let $p_i = \mathbb{P}\{M_u^E(J) = i\}$, then $\mathbb{P}\{M_u^E(J) \geq 1\} = \sum_{i=1}^{\infty} p_i$ and $\mathbb{E}\{M_u^E(J)\} = \sum_{i=1}^{\infty} i p_i$, it follows that

$$\mathbb{E}\{M_u^E(J)\} - \mathbb{P}\{M_u^E(J) \geq 1\} = \sum_{i=2}^{\infty} (i-1)p_i$$
$$\leq \sum_{i=2}^{\infty} \frac{i(i-1)}{2} p_i = \frac{1}{2}\mathbb{E}\{M_u^E(J)(M_u^E(J) - 1)\}.$$

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Together with the obvious bound $\mathbb{P}\{M_u^E(J) \geq 1, M_u^E(J') \geq 1\} \leq \mathbb{E}\{M_u^E(J)M_u^E(J')\}$, we obtain the following lower bound for the excursion probability,

$$\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} \geq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \left(\mathbb{E}\{M_u^E(J)\} - \frac{1}{2}\mathbb{E}\{M_u^E(J)(M_u^E(J) - 1)\}\right)$$
$$- \sum_{J \neq J'} \mathbb{E}\{M_u^E(J)M_u^E(J')\}. \tag{4.2}$$

Define the number of *local maxima* above level u as

$$M_u(J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \text{index}(\nabla^2 X_{|J}(t)) = k\},$$

then obviously $M_u(J) \geq M_u^E(J)$, and $M_u(J)$ is the same as $\widetilde{\mu}_k(J)$ defined in (3.3) with $i = k$. It follows similarly that

$$\sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u(J)\} \geq \mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\}$$
$$\geq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \left(\mathbb{E}\{M_u(J)\} - \frac{1}{2}\mathbb{E}\{M_u(J)(M_u(J) - 1)\}\right) - \sum_{J \neq J'} \mathbb{E}\{M_u(J)M_u(J')\}. \tag{4.3}$$

We will use (4.1) and (4.2) to estimate the excursion probability for the general case, see Theorem 4.8. Inequalities in (4.3) provide another method to approximate the excursion probability in some special cases, see Theorem 4.7. The advantage of (4.3) is that the principal term induced by $\sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u(J)\}$ is much easier to compute compared with the one induced by $\sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u^E(J)\}$.

4.2 Estimating the Moments: Major Terms and Error Terms

The following two lemmas provide the estimations for the principal terms in approximating the excursion probability.

Lemma 4.2 *Let X be a Gaussian field as in Theorem 3.2. Then for each $J \in \partial_k T$ with $k \geq 1$, there exists some constant $\alpha > 0$ such that*

$$\mathbb{E}\{M_u(J)\} = \frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}} \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\Big(\frac{u}{\theta_t}\Big) e^{-u^2/(2\theta_t^2)} \, dt(1 + o(e^{-\alpha u^2})). \tag{4.4}$$

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Proof   Following the notation in the proof of Lemma 3.1, we obtain similarly that

$$\mathbb{E}\{M_u(J)\} = \int_J p_{\nabla X_{|J}(t)}(0)dt \, \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_k\}} \mathbb{1}_{\{X(t) \geq u\}} | \nabla X_{|J}(t) = 0\}$$

$$= \int_J dt \int_u^\infty dx \frac{(-1)^k e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t} \mathbb{E}\{\det \nabla^2 X_{|J}(t) \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_k\}} | X(t) = x, \nabla X_{|J}(t) = 0\}.$$

$$(4.5)$$

Recall $\nabla^2 X_{|J}(t) = Q_t^{-1} Q_t \nabla^2 X_{|J}(t) Q_t Q_t^{-1}$ and we can write (3.12) as

$$\mathbb{E}\{Q_t \nabla^2 X_{|J}(t) Q_t | X(t) = x, \nabla X_{|J}(t) = 0\} = -\frac{x}{\theta_t^2} I_k.$$

Make change of variables

$$V(t) = Q_t \nabla^2 X_{|J}(t) Q_t + \frac{x}{\theta_t^2} I_k,$$

where $V(t) = (V_{ij}(t))_{1 \leq i,j \leq k}$. Then $(V(t)|X(t) = x, \nabla X_{|J}(t) = 0)$ is a Gaussian matrix whose mean is 0 and covariance is the same as that of $(Q_t \nabla^2 X_{|J}(t) Q_t | X(t) = x, \nabla X_{|J}(t) = 0)$. Denote the density of Gaussian vectors $((V_{ij}(t))_{1 \leq i \leq j \leq k} | X(t) = x, \nabla X_{|J}(t) = 0)$ by $h_t(v)$, $v = (v_{ij})_{1 \leq i \leq j \leq k} \in \mathbb{R}^{k(k+1)/2}$, then

$$\mathbb{E}\{\det(Q_t \nabla^2 X_{|J}(t) Q_t) \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_k\}} | X(t) = x, \nabla X_{|J}(t) = 0\}$$

$$= \mathbb{E}\{\det(Q_t \nabla^2 X_{|J}(t) Q_t) \mathbb{1}_{\{Q_t \nabla^2 X_{|J}(t) Q_t \in \mathcal{D}_k\}} | X(t) = x, \nabla X_{|J}(t) = 0\} \qquad (4.6)$$

$$= \int_{v:(v_{ij}) - \frac{x}{\theta_t^2} I_k \in \mathcal{D}_k} \det\left((v_{ij}) - \frac{x}{\theta_t^2} I_k\right) h_t(v) \, dv,$$

where (v_{ij}) is the abbreviation of matrix $(v_{ij})_{1 \leq i,j \leq k}$. Since $\{\theta_t^2 : t \in T\}$ is bounded, there exists a constant $c > 0$ such that

$$(v_{ij}) - \frac{x}{\theta_t^2} I_k \in \mathcal{D}_k, \quad \forall \|(v_{ij})\| := \left(\sum_{i,j=1}^k v_{ij}^2\right)^{1/2} < \frac{x}{c}.$$

Thus we can write (4.6) as

$$\int_{\mathbb{R}^{k(k+1)/2}} \det\left((v_{ij}) - \frac{x}{\theta_t^2} I_k\right) h_t(v) dv - \int_{v:(v_{ij}) - \frac{x}{\theta_t^2} I_k \notin \mathcal{D}_k} \det\left((v_{ij}) - \frac{x}{\theta_t^2} I_k\right) h_t(v) \, dv$$

$$= \mathbb{E}\{\det(Q_t \nabla^2 X_{|J}(t) Q_t) | X(t) = x, \nabla X_{|J}(t) = 0\} + Z(t, x),$$

$$(4.7)$$

where $Z(t, x)$ is the second integral in the first line of (4.7) and it satisfies

$$|Z(t, x)| \leq \int_{\|(v_{ij})\| \geq \frac{x}{c}} \left|\det\left((v_{ij}) - \frac{x}{\theta_t^2} I_k\right)\right| h_t(v) dv.$$

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Denote by $G(t)$ the covariance matrix of $((V_{ij}(t))_{1 \le i \le j \le k} | X(t) = x, \nabla X_{|J}(t) = 0)$, then by Lemma 6.2 in the Appendix, the eigenvalues of $G(t)$ are bounded from above and below by positive and finite constants for all $t \in T$, hence so are those of $(G(t))^{-1}$ for all $t \in T$. It follows that there exists some constant $\alpha' > 0$ such that $h_t(v) = o(e^{-\alpha' \|(v_{ij})\|^2})$ and hence $|Z(t, x)| = o(e^{-\alpha x^2})$ for some constant $\alpha > 0$ uniformly for all $t \in T$. Combine this with (4.5), (4.6), (4.7) and the proof of Lemma 3.1, yielding the result. $\qquad \square$

Lemma 4.3 *Let X be a Gaussian field as in Theorem 3.2. Then for each $J \in \partial_k T$ with $k \ge 1$, there exists some constant $\alpha > 0$ such that*

$$
\begin{aligned}
\mathbb{E}\{M_u^E(J)\} = {} & \frac{1}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} \int_J dt \int_u^\infty dx \int \cdots \int_{E(J)} dy_{J_1} \cdots dy_{J_{N-k}} \\
& \times \frac{|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k} H_k\left(\frac{x}{\gamma_t} + \gamma_t C_{J_1}(t)y_{J_1} + \cdots + \gamma_t C_{J_{N-k}}(t)y_{J_{N-k}}\right) \\
& \times p_{X(t), X_{J_1}(t), \ldots, X_{J_{N-k}}(t)}(x, y_{J_1}, \ldots, y_{J_{N-k}} | \nabla X_{|J}(t) = 0)(1 + o(e^{-\alpha u^2})).
\end{aligned}
\tag{4.8}
$$

Proof   Similarly to the proof in Theorem 3.2, we see that $\mathbb{E}\{M_u^E(J)\}$ is equal to

$$
\begin{aligned}
& \int_J \frac{(-1)^k|\Lambda_J - \Lambda_J(t)|}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} dt \int_u^\infty dx \int \cdots \int_{E(J)} dy_{J_1} \cdots dy_{J_{N-k}} \mathbb{E}\{\det(Q_t \nabla^2 X_{|J}(t) Q_t) \\
& \quad \times \mathbb{1}_{\{Q_t \nabla^2 X_{|J}(t) Q_t \in \mathcal{D}_k\}} | X(t) = x, X_{J_1}(t) = y_{J_1}, \ldots, X_{J_{N-k}}(t) = y_{J_{N-k}}, \nabla X_{|J}(t) = 0\} \\
& \quad \times p_{X(t), X_{J_1}(t), \ldots, X_{J_{N-k}}(t)}(x, y_{J_1}, \ldots, y_{J_{N-k}} | \nabla X_{|J}(t) = 0) \\
:= {} & \int_J \frac{(-1)^k|\Lambda_J - \Lambda_J(t)|}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} dt \int_u^\infty dx \, K(t, x),
\end{aligned}
$$

where Q_t is the positive definite matrix in (3.10). Then using similar argument in the proof of Lemma 4.2 to estimate $K(t, x)$, we obtain the desired result. $\qquad \square$

Denote by $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ the n-dimensional unit sphere. We call a function $h(u)$ *super-exponentially small* [when compared with $\mathbb{P}(\sup_{t \in T} X(t) \ge u)$], if there exists a constant $\alpha > 0$ such that $h(u) = o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)})$ as $u \to \infty$.

The following lemma is Lemma 4 in Piterbarg (1996b). It shows that the factorial moments are usually super-exponentially small.

Lemma 4.4 *Let $\{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian field satisfying (**H**1) and (**H**3). Then for any $\varepsilon > 0$, there exists $\varepsilon_1 > 0$ such that for any $J \in \partial_k T$ and u large enouth,*

$$
\mathbb{E}\{M_u(J)(M_u(J) - 1)\} \le e^{-u^2/(2\beta_J^2 + \varepsilon)} + e^{-u^2/(2\sigma_J^2 - \varepsilon_1)},
$$

where $\beta_J^2 = \sup_{t \in J} \sup_{e \in \mathbb{S}^{k-1}} \mathrm{Var}(X(t) | \nabla X_{|J}(t), \nabla^2 X_{|J}(t)e)$ and $\sigma_J^2 = \sup_{t \in J} \mathrm{Var}(X(t))$.

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Corollary 4.5 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments satisfying* (**H1**), (**H2**) *and* (**H3**). *Then for all $J \in \partial_k T$, $\mathbb{E}\{M_u(J)(M_u(J) - 1)\}$ and $\mathbb{E}\{M_u^E(J)(M_u^E(J) - 1)\}$ are super-exponentially small.*

Proof Since $M_u^E(J) \leq M_u(J)$, we only need to show that $\mathbb{E}\{M_u(J)(M_u(J) - 1)\}$ is super-exponentially small. If $k = 0$, then $M_u(J)$ is either 0 or 1 and hence $\mathbb{E}\{M_u(J)(M_u(J)-1)\} = 0$. If $k \geq 1$, then, thanks to Lemma 4.4, it suffices to show that β_J^2 is strictly less than σ_T^2.

Clearly, $\text{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e) \leq \sigma_T^2$. Applying Lemma 6.1 yields that

$$\text{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e) = \sigma_T^2 \Rightarrow \mathbb{E}\{X(t)(\nabla^2 X_{|J}(t)e)\} = 0.$$

Note that the right hand side above is equivalent to $(\Lambda_J(t) - \Lambda_J)e = 0$. By (**H2**), $\Lambda_J(t) - \Lambda_J$ is negative definite, which implies $(\Lambda_J(t) - \Lambda_J)e \neq 0$ for all $e \in \mathbb{S}^{k-1}$, so that

$$\sup_{e \in \mathbb{S}^{k-1}} \text{Var}(X(t)|\nabla_{|J}X(t), \nabla_{|J}^2 X(t)e) < \sigma_T^2.$$

Therefore $\beta_J^2 < \sigma_T^2$ by continuity. $\qquad\square$

The following lemma shows that the cross terms in (4.2) and (4.3) are super-exponentially small if the two faces are not adjacent. For the case when the faces are adjacent, the proof is more technical, see the proofs in Theorems 4.7 and 4.8.

Lemma 4.6 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments satisfying* (**H1**) *and* (**H3**). *Let J and J' be two faces of T such that their distance is positive, i.e., $\inf_{t \in J, s \in J'} \|s - t\| > \delta_0$ for some $\delta_0 > 0$, then $\mathbb{E}\{M_u(J)M_u(J')\}$ is super-exponentially small.*

Proof We first consider the case when $\dim(J) = k \geq 1$ and $\dim(J') = k' \geq 1$. By the Kac-Rice metatheorem for higher moments [the proof is the same as that of Theorem 11.5.1 in Adler and Taylor (2007)],

$$
\begin{aligned}
\mathbb{E}\{M_u(J)M_u(J')\} &= \int_J dt \int_{J'} ds\, \mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)|\mathbb{1}_{\{X(t)\geq u, X(s)\geq u\}} \\
&\quad \times \mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k, \nabla^2 X_{|J'}(s)\in\mathcal{D}_{k'}\}}|X(t) = x, X(s) = y, \nabla X_{|J}(t) = 0, \\
&\quad \nabla X_{|J'}(s) = 0\}p_{X(t),X(s),\nabla X_{|J}(t),\nabla X_{|J'}(s)}(x,y,0,0) \\
&\leq \int_J dt \int_{J'} ds \int_u^\infty dx \int_u^\infty dy\, \mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)| \\
&\quad |X(t) = x, X(s) = y, \nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0\}p_{X(t),X(s)}(x,y) \\
&\quad \times p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0|X(t) = x, X(s) = y).
\end{aligned}
\tag{4.9}
$$

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Note that the following two inequalities hold: for constants a_i and b_j,

$$\prod_{i=1}^{k} |a_i| \prod_{j=1}^{k'} |b_j| \leq \frac{1}{k+k'}\left(\sum_{i=1}^{k} |a_i|^{k+k'} + \sum_{j=1}^{k'} |b_j|^{k+k'}\right);$$

and for any Gaussian variable ξ and positive integer l,

$$\mathbb{E}|\xi|^l \leq \mathbb{E}(|\mathbb{E}\xi| + |\xi - \mathbb{E}\xi|)^l \leq 2^l(|\mathbb{E}\xi|^l + \mathbb{E}|\xi - \mathbb{E}\xi|^l) \leq 2^l(|\mathbb{E}\xi|^l + C_l(\mathrm{Var}(\xi))^{l/2}),$$

where the constant C_l depends only on l. Combining these two inequalities with Lemma 6.1, we get that there exist some positive constants C_1 and N_1 such that for large x and y,

$$\sup_{t \in J, s \in J'} \mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)||X(t) = x, X(s) = y,$$
$$\nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0\} \leq C_1 x^{N_1} y^{N_1}. \tag{4.10}$$

Also, there exists a positive constant C_2 such that

$$\sup_{t \in J, s \in J'} p_{\nabla X_{|J}(t), \nabla X_{|J'}(s)}(0, 0|X(t) = x, X(s) = y)$$
$$\leq \sup_{t \in J, s \in J'} (2\pi)^{-(k+k')/2}[\det\mathrm{Cov}(\nabla X_{|J}(t), \nabla X_{|J'}(s)|X(t) = x, X(s) = y)]^{-1/2} \leq C_2. \tag{4.11}$$

Let $\rho(\delta_0) = \sup_{\|s-t\|>\delta_0} \frac{\mathbb{E}\{X(t)X(s)\}}{\sigma_t \sigma_s}$ which is strictly less than 1 due to (**H3**), then $\forall \varepsilon > 0$, there exists a positive constant C_3 such that for all $t \in J$, $s \in J'$ and u large enough,

$$\int_u^\infty \int_u^\infty x^{N_1} y^{N_1} p_{X(t), X(s)}(x, y) dx dy = \mathbb{E}\{[X(t)X(s)]^{N_1} \mathbb{1}_{\{X(t) \geq u, X(s) \geq u\}}\}$$
$$\leq \mathbb{E}\{[X(t) + X(s)]^{2N_1} \mathbb{1}_{\{X(t)+X(s) \geq 2u\}}\} \leq C_3 \exp\left(\varepsilon u^2 - \frac{u^2}{(1+\rho(\delta_0))\sigma_T^2}\right). \tag{4.12}$$

Combine (4.9) with (4.10), (4.11) and (4.12), yielding that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small.

When only one of the faces, say J, is a singleton, then let $J = \{t_0\}$ and we have

$$\mathbb{E}\{M_u(J)M_u(J')\} \leq \int_{J'} ds \int_u^\infty dx \int_u^\infty dy\, p_{X(t_0), X(s), \nabla X_{|J'}(s)}(x, y, 0)$$
$$\times \mathbb{E}\{|\det\nabla^2 X_{|J'}(s)||X(t_0) = x, X(s) = y, \nabla X_{|J'}(s) = 0\}. \tag{4.13}$$

Following the previous discussion yields that $\mathbb{E}\{M_u(J)M_u(J')\}$ is super-exponentially small.

Finally, if both J and J' are singletons, then $\mathbb{E}\{M_u(J)M_u(J')\}$ becomes the joint probability of two Gaussian variables exceeding level u and hence is trivial. $\square$

4.3 Main Results and Their Proofs

Theorem 4.7 *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments such that (**H**1), (**H**2) and (**H**3) are fulfilled. Suppose that for any face J,*

$$\{t \in J : \nu(t) = \sigma_T^2, \nu_j(t) = 0 \text{ for some } j \notin \sigma(J)\} = \emptyset. \tag{4.14}$$

Then there exists some constant $\alpha > 0$ such that

$$\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} = \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u(J)\} + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)})$$

$$= \sum_{\{t\} \in \partial_0 T} \Psi\Big(\frac{u}{\sigma_t}\Big) + \sum_{k=1}^{N} \sum_{J \in \partial_k T} \frac{1}{(2\pi)^{(k+1)/2} |\Lambda_J|^{1/2}} \tag{4.15}$$

$$\times \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\Big(\frac{u}{\theta_t}\Big) e^{-u^2/(2\theta_t^2)} \, dt + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}).$$

Proof Since the second equality in (4.15) follows from Lemma 4.2 directly, we only need to prove the first one. By (4.3) and Corollary 4.5, it suffices to show that the last term in (4.3) is super-exponentially small. Thanks to Lemma 4.6, we only need to consider the case when the distance of J and J' is 0, or $I := \bar{J} \cap \bar{J'} \neq \emptyset$. Without loss of generality, assume

$$\sigma(J) = \{1, \ldots, m, m+1, \ldots, k\}, \quad \sigma(J') = \{1, \ldots, m, k+1, \ldots, k+k'-m\}, \tag{4.16}$$

where $0 \leq m \leq k \leq k' \leq N$ and $k' \geq 1$. If $k = 0$, we consider $\sigma(J) = \emptyset$ by convention. Under such assumption, $J \in \partial_k T$, $J' \in \partial_{k'} T$ and $\dim(I) = m$.

Case 1: $k = 0$, i.e. J is a singleton, say $J = \{t_0\}$. If $\nu(t_0) < \sigma_T^2$, then by (4.13), it is trivial to show that $\mathbb{E}\{M_u(J)M_u(J')\}$ is super-exponentially small. Now we consider the case $\nu(t_0) = \sigma_T^2$. Due to (4.14), $\mathbb{E}\{X(t_0)X_1(t_0)\} \neq 0$ and hence by continuity, there exists $\delta > 0$ such that $\mathbb{E}\{X(s)X_1(s)\} \neq 0$ for all $\|s - t_0\| \leq \delta$. It follows from (4.13) that $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded from above by

$$\int_{s \in J' : \|s-t_0\| > \delta} ds \int_u^\infty dx \int_u^\infty dy \, \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)| \, | \, X(t_0) = x, X(s) = y, \nabla X_{|J'}(s) = 0\}$$

$$\times p_{X(t_0), X(s), \nabla X_{|J'}(s)}(x, y, 0)$$

$$+ \int_{s \in J' : \|s-t_0\| \leq \delta} ds \int_u^\infty dy \, \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)| \, | \, X(s) = y, \nabla X_{|J'}(s) = 0\} p_{X(s), \nabla X_{|J'}(s)}(y, 0)$$

$$:= I_1 + I_2.$$

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Following the proof of Lemma 4.6 yields that I_1 is super-exponentially small. We apply Lemma 6.1 to obtain that there exists $\varepsilon_0 > 0$ such that

$$\sup_{s \in J': \|s - t_0\| \leq \delta} \mathrm{Var}(X(s)|\nabla X_{|J'}(s)) \leq \sup_{s \in J': \|s - t_0\| \leq \delta} \mathrm{Var}(X(s)|X_1(s)) \leq \sigma_T^2 - \varepsilon_0.$$

Then I_2 and hence $\mathbb{E}\{M_u(J)M_u(J')\}$ are super-exponentially small.

Case 2: $k \geq 1$. For all $t \in I$ with $\nu(t) = \sigma_T^2$, by assumption (4.14), $\mathbb{E}\{X(t)X_i(t)\} \neq 0$, $\forall\, i = m+1, \ldots, k+k'-m$. Note that I is a compact set, by Lemma 6.1 and the uniform continuity of conditional variance, there exist $\varepsilon_1, \delta_1 > 0$ such that

$$\sup_{t \in B, s \in B'} \mathrm{Var}(X(t)|X_{m+1}(t), \ldots, X_k(t), X_{k+1}(s), \ldots, X_{k+k'-m}(s)) \leq \sigma_T^2 - \varepsilon_1, \qquad (4.17)$$

where $B = \{t \in J : \mathrm{dist}(t, I) \leq \delta_1\}$ and $B' = \{s \in J' : \mathrm{dist}(s, I) \leq \delta_1\}$. It follows from (4.9) that $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded by

$$\int\!\!\int_{(J \times J') \backslash (B \times B')} dt\,ds \int_u^\infty dx \int_u^\infty dy\, p_{X(t), X(s), \nabla X_{|J}(t), \nabla X_{|J'}(s)}(x, y, 0, 0)$$
$$\times \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)|\,|\,X(t) = x, X(s) = y, \nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0\}$$
$$+ \int\!\!\int_{B \times B'} dt\,ds \int_u^\infty dx\, p_{X(t)}(x|\nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0) p_{\nabla X_{|J}(t), \nabla X_{|J'}(s)}(0, 0)$$
$$\times \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)|\,|\,X(t) = x, \nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0\}$$
$$:= I_3 + I_4.$$

Note that

$$(J \times J') \backslash (B \times B') = \Big((J \backslash B) \times B'\Big) \bigcup \Big(B \times (J \backslash B)\Big) \bigcup \Big((J \backslash B) \times (J \backslash B)\Big). \qquad (4.18)$$

Since each product set on the right hand side of (4.18) consists of two sets with positive distance, following the proof of Lemma 4.6 yields that I_3 is super-exponentially small.

For I_4, taking into account (4.17), one has

$$\sup_{t \in B, s \in B'} \mathrm{Var}\big(X(t)|\nabla X_{|J}(t), \nabla X_{|J'}(s)\big) \leq \sigma_T^2 - \varepsilon_1. \qquad (4.19)$$

To estimate

$$p_{\nabla X_{|J}(t), \nabla X_{|J'}(s)}(0, 0) = (2\pi)^{-(k+k')/2} (\det \mathrm{Cov}(\nabla X_{|J}(t), \nabla X_{|J'}(s)))^{-1/2}, \qquad (4.20)$$

we write the determinant on the right hand side of (4.20) as

$$\mathrm{detCov}(X_{m+1}(t), \ldots, X_k(t), X_{k+1}(s), \ldots, X_{k+k'-1}(s)|X_1(t), \ldots, X_m(t), X_1(s), \ldots, X_m(s))$$
$$\times \mathrm{detCov}(X_1(t), \ldots, X_m(t), X_1(s), \ldots, X_m(s)),$$

$$(4.21)$$

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where the first determinant in (4.21) is bounded away from zero due to (**H3**). By (**H1**), as shown in Piterbarg (1996b), applying Taylor's formula, we can write

$$\nabla X(s) = \nabla X(t) + \nabla^2 X(t)(s - t)^T + \|s - t\|^{1+\eta} Y_{t,s}, \tag{4.22}$$

where $Y_{t,s} = (Y_{t,s}^1, \ldots, Y_{t,s}^N)^T$ is a Gaussian vector field with bounded variance uniformly for all $t \in J$, $s \in J'$. Hence as $\|s - t\| \to 0$, the second determinant in (4.21) becomes

$$\begin{aligned}
&\det \mathrm{Cov}(X_1(t), \ldots, X_m(t), X_1(t) + \langle \nabla X_1(t), s - t \rangle + \|s - t\|^{1+\eta} Y_{t,s}^1, \ldots, \\
&\qquad X_m(t) + \langle \nabla X_m(t), s - t \rangle + \|s - t\|^{1+\eta} Y_{t,s}^m) \\
&= \det \mathrm{Cov}(X_1(t), \ldots, X_m(t), \langle \nabla X_1(t), s - t \rangle + \|s - t\|^{1+\eta} Y_{t,s}^1, \ldots, \\
&\qquad \langle \nabla X_m(t), s - t \rangle + \|s - t\|^{1+\eta} Y_{t,s}^m) \\
&= \|s - t\|^{2m} \det \mathrm{Cov}(X_1(t), \ldots, X_m(t), \langle \nabla X_1(t), e_{t,s} \rangle, \ldots, \langle \nabla X_m(t), e_{t,s} \rangle)(1 + o(1)),
\end{aligned} \tag{4.23}$$

where $e_{t,s} = (s - t)^T / \|s - t\|$ and due to (**H3**), the last determinant in (4.23) is bounded away from zero uniformly for all $t \in J$ and $s \in J'$. It then follows from (4.21) and (4.23) that

$$\det \mathrm{Cov}(\nabla X_{|J}(t), \nabla X_{|J'}(s)) \geq C_1 \|s - t\|^{2m} \tag{4.24}$$

for some constant $C_1 > 0$. Similarly to (4.10), there exist constants $C_2, N_1 > 0$ such that

$$\begin{aligned}
&\sup_{t \in J, s \in J'} \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)| \,|\, X(t) = x, \nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0\} \\
&\quad \leq C_2(1 + x^{N_1}).
\end{aligned} \tag{4.25}$$

Combining (4.19) with (4.20), (4.24) and (4.25), and noting that $m < k'$ implies $1/\|s - t\|^m$ is integrable on $J \times J'$, we conclude that I_4 and hence $\mathbb{E}\{M_u(J)M_u(J')\}$ are finite and super-exponentially small. $\qquad\square$

Theorem 4.8 *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments such that (**H1**), (**H2**) and (**H3**) are fulfilled. Then there exists some constant $\alpha > 0$ such that*

$$\begin{aligned}
\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} &= \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u^E(J)\} + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}) \\
&= \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}),
\end{aligned} \tag{4.26}$$

where $\mathbb{E}\{\varphi(A_u)\}$ is formulated in Theorem 3.2.

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It is worth mentioning here that the main idea for the proof of Theorem 4.8 comes from Azaïs and Delmas (2002) [especially Theorem 4]. Before showing the proof, we list the following two lemmas whose proofs are given in the Appendix.

Lemma 4.9 *Under* (**H2**), *there exists a constant* $\alpha_0 > 0$ *such that*

$$\langle e, (\Lambda - \Lambda(t))e \rangle \geq \alpha_0, \quad \forall\, t \in T, e \in \mathbb{S}^{N-1}.$$

Lemma 4.10 *Let* $\{\xi_1(t) : t \in T_1\}$ *and* $\{\xi_2(t) : t \in T_2\}$ *be two Gaussian random fields. Let*

$$\sigma_i^2(t) = \mathrm{Var}(\xi_i(t)), \quad \overline{\sigma}_i = \sup_{t \in T_i} \sigma_i(t), \quad \underline{\sigma}_i = \inf_{t \in T_i} \sigma_i(t),$$

$$\rho(t, s) = \frac{\mathbb{E}\{\xi_1(t)\xi_2(s)\}}{\sigma_1(t)\sigma_2(s)}, \quad \overline{\rho} = \sup_{t \in T_1, s \in T_2} \rho(t, s), \quad \underline{\rho} = \inf_{t \in T_1, s \in T_2} \rho(t, s),$$

and assume $0 < \underline{\sigma}_i \leq \overline{\sigma}_i < \infty$, *where* $i = 1, 2$. *If* $0 < \underline{\rho} \leq \overline{\rho} < 1$, *then for any* $N_1, N_2 > 0$, *there exists some* $\alpha > 0$ *such that as* $u \to \infty$,

$$\sup_{t \in T_1, s \in T_2} \mathbb{E}\{(1 + |\xi_1(t)|^{N_1} + |\xi_2(s)|^{N_2}) \mathbb{1}_{\{\xi_1(t) \geq u, \xi_2(s) < 0\}}\} = o(e^{-\alpha u^2 - u^2/(2\overline{\sigma}_1^2)}).$$

Similarly, if $-1 < \underline{\rho} \leq \overline{\rho} < 0$, *then*

$$\sup_{t \in T_1, s \in T_2} \mathbb{E}\{(1 + |\xi_1(t)|^{N_1} + |\xi_2(s)|^{N_2}) \mathbb{1}_{\{\xi_1(t) \geq u, \xi_2(s) > 0\}}\} = o(e^{-\alpha u^2 - u^2/(2\overline{\sigma}_1^2)}).$$

Proof of Theorem 4.8 Note that the second equality in (4.26) follows from Theorem 3.2 and Lemma 4.3, and similarly to the proof in Theorem 4.7, we only need to show that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small when J and J' are neighboring. Let $I := \bar{J} \cap \bar{J}' \neq \emptyset$. We follow the assumptions in (4.16) and assume also that all elements in $\varepsilon(J)$ and $\varepsilon(J')$ are 1, which implies $E(J) = \mathbb{R}_+^{N-k}$ and $E(J') = \mathbb{R}_+^{N-k'}$.

We first consider the case $k \geq 1$. By the Kac-Rice metatheorem, $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded from above by

$$\int_J dt \int_{J'} ds \int_u^\infty dx \int_u^\infty dy \int_0^\infty dz_{k+1} \cdots \int_0^\infty dz_{k+k'-m} \int_0^\infty dw_{m+1} \cdots \int_0^\infty dw_k$$

$$\mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)| \,|\, X(t) = x, X(s) = y, \nabla X_{|J}(t) = 0, X_{k+1}(t) = z_{k+1},$$

$$\dots, X_{k+k'-m}(t) = z_{k+k'-m}, \nabla X_{|J'}(s) = 0, X_{m+1}(s) = w_{m+1}, \dots, X_k(s) = w_k\} \quad (4.27)$$

$$\times\, p_{t,s}(x, y, 0, z_{k+1}, \dots, z_{k+k'-m}, 0, w_{m+1}, \dots, w_k)$$

$$:= \int\!\!\int_{J \times J'} A(t, s)\, dt ds,$$

where $p_{t,s}(x, y, 0, z_{k+1}, \ldots, z_{k+k'-m}, 0, w_{m+1}, \ldots, w_k)$ is the density of

$$(X(t), X(s), \nabla X_{|J}(t), X_{k+1}(t), \ldots, X_{k+k'-m}(t), \nabla X_{|J'}(s), X_{m+1}(s), \ldots, X_k(s))$$

evaluated at $(x, y, 0, z_{k+1}, \ldots, z_{k+k'-m}, 0, w_{m+1}, \ldots, w_k)$.

Let $\{e_1, e_2, \ldots, e_N\}$ be the standard orthonormal basis of $\mathbb{R}^N$. For $t \in J$ and $s \in J'$, let $e_{t,s} = (s - t)^T / \|s - t\|$ and let $\alpha_i(t, s) = \langle e_i, (\Lambda - \Lambda(t))e_{t,s}\rangle$, then

$$(\Lambda - \Lambda(t))e_{t,s} = \sum_{i=1}^{N} \langle e_i, (\Lambda - \Lambda(t))e_{t,s}\rangle e_i = \sum_{i=1}^{N} \alpha_i(t, s)e_i. \tag{4.28}$$

By Lemma 4.9, there exists some $\alpha_0 > 0$ such that

$$\langle e_{t,s}, (\Lambda - \Lambda(t))e_{t,s}\rangle \geq \alpha_0 \tag{4.29}$$

for all t and s. Under the assumptions (4.16) and that all elements in $\varepsilon(J)$ and $\varepsilon(J')$ are 1, we have the following representation,

$$t = (t_1, \ldots, t_m, t_{m+1}, \ldots, t_k, b_{k+1}, \ldots, b_{k+k'-m}, 0, \ldots, 0),$$
$$s = (s_1, \ldots, s_m, b_{m+1}, \ldots, b_k, s_{k+1}, \ldots, s_{k+k'-m}, 0, \ldots, 0),$$

where $t_i \in (a_i, b_i)$ for all $i \in \sigma(J)$ and $s_j \in (a_j, b_j)$ for all $j \in \sigma(J')$. Therefore,

$$
\begin{aligned}
\langle e_i, e_{t,s}\rangle &\geq 0, &&\forall\, m + 1 \leq i \leq k, \\
\langle e_i, e_{t,s}\rangle &\leq 0, &&\forall\, k + 1 \leq i \leq k + k' - m, \\
\langle e_i, e_{t,s}\rangle &= 0, &&\forall\, k + k' - m < i \leq N.
\end{aligned}
\tag{4.30}
$$

Let

$$
\begin{aligned}
D_i &= \{(t, s) \in J \times J' : \alpha_i(t, s) \geq \beta_i\}, &&\text{if } m + 1 \leq i \leq k, \\
D_i &= \{(t, s) \in J \times J' : \alpha_i(t, s) \leq -\beta_i\}, &&\text{if } k + 1 \leq i \leq k + k' - m, \\
D_0 &= \left\{(t, s) \in J \times J' : \sum_{i=1}^{m} \alpha_i(t, s)\langle e_i, e_{t,s}\rangle \geq \beta_0\right\},
\end{aligned}
\tag{4.31}
$$

where $\beta_0, \beta_1, \ldots, \beta_{k+k'-m}$ are positive constants such that $\beta_0 + \sum_{i=m+1}^{k+k'-m} \beta_i < \alpha_0$. It follows from (4.30) and (4.31) that, if (t, s) does not belong to any of $D_0, D_m, \ldots, D_{k+k'-m}$, then by (4.28),

$$\langle (\Lambda - \Lambda(t))e_{t,s}, e_{t,s}\rangle = \sum_{i=1}^{N} \alpha_i(t, s)\langle e_i, e_{t,s}\rangle \leq \beta_0 + \sum_{i=m+1}^{k+k'-m} \beta_i < \alpha_0,$$

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which contradicts (4.29). Thus $D_0 \cup \cup_{i=m+1}^{k+k'-m} D_i$ is a covering of $J \times J'$, by (4.27),

$$\mathbb{E}\{M_u^E(J)M_u^E(J')\} \leq \int\int_{D_0} A(t,s)\,dtds + \sum_{i=m+1}^{k+k'-m} \int\int_{D_i} A(t,s)\,dtds.$$

We first show that $\int\int_{D_0} A(t,s)\,dtds$ is super-exponentially small. Similarly to the proof of Theorem 4.7, applying (4.20), (4.24) and (4.25), we obtain

$$\begin{aligned}
&\int\int_{D_0} A(t,s)\,dtds \\
&\leq \int\int_{D_0} dtds \int_u^\infty dx\, p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0)p_{X(t)}(x|\nabla X_{|J}(t)=0,\nabla X_{|J'}(s)=0) \\
&\qquad \times \mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)||X(t)=x,\nabla X_{|J}(t)=0,\nabla X_{|J'}(s)=0\} \\
&\leq C_1' \int\int_{D_0} dtds \int_u^\infty dx(1+x^{N_1})\|s-t\|^{-m}p_{X(t)}(x|\nabla X_{|J}(t)=0,\nabla X_{|J'}(s)=0),
\end{aligned} \tag{4.32}$$

for some positive constants C_1' and N_1. Due to Lemma 4.6, we only need to consider the case when $\|s-t\|$ is small. It follows from Taylor's formula (4.22) that as $\|s-t\| \to 0$,

$$\begin{aligned}
&\mathrm{Var}(X(t)|\nabla X_{|J}(t),\nabla X_{|J'}(s)) \leq \mathrm{Var}(X(t)|X_1(t),\ldots,X_m(t),X_1(s),\ldots,X_m(s)) \\
&= \mathrm{Var}(X(t)|X_1(t),\ldots,X_m(t),X_1(t)+\langle\nabla X_1(t),s-t\rangle+\|s-t\|^{1+\eta}Y_{t,s}^1,\ldots, \\
&\qquad\qquad X_m(t)+\langle\nabla X_m(t),s-t\rangle+\|s-t\|^{1+\eta}Y_{t,s}^m) \\
&= \mathrm{Var}(X(t)|X_1(t),\ldots,X_m(t),\langle\nabla X_1(t),e_{t,s}\rangle+\|s-t\|^\eta Y_{t,s}^1,\ldots, \\
&\qquad\qquad \langle\nabla X_m(t),e_{t,s}\rangle+\|s-t\|^\eta Y_{t,s}^m) \\
&\leq \mathrm{Var}(X(t)|\langle\nabla X_1(t),e_{t,s}\rangle+\|s-t\|^\eta Y_{t,s}^1,\ldots,\langle\nabla X_m(t),e_{t,s}\rangle+\|s-t\|^\eta Y_{t,s}^m) \\
&= \mathrm{Var}(X(t)|\langle\nabla X_1(t),e_{t,s}\rangle,\ldots,\langle\nabla X_m(t),e_{t,s}\rangle) + o(1).
\end{aligned} \tag{4.33}$$

By Lemma 6.2, the eigenvalues of $[\mathrm{Cov}(\langle\nabla X_1(t),e_{t,s}\rangle,\ldots,\langle\nabla X_m(t),e_{t,s}\rangle)]^{-1}$ are bounded uniformly in t and s. Note that $\mathbb{E}\{X(t)\langle\nabla X_i(t),e_{t,s}\rangle\} = -\alpha_i(t,s)$. Applying these facts and Lemma 6.1 to the last line of (4.33), we see that there exist constants $C_2 > 0$ and $\varepsilon_0 > 0$ such that for $\|s-t\|$ sufficiently small,

$$\mathrm{Var}(X(t)|\nabla X_{|J}(t),\nabla X_{|J'}(s)) \leq \sigma_T^2 - C_2 \sum_{i=1}^m \alpha_i^2(t,s) + o(1) < \sigma_T^2 - \varepsilon_0, \tag{4.34}$$

where the last inequality is due to the fact that $(t,s) \in D_0$ implies

$$\sum_{i=1}^m \alpha_i^2(t,s) \geq \sum_{i=1}^m \alpha_i^2(t,s)|\langle e_i,e_{t,s}\rangle|^2 \geq \frac{1}{m}\left(\sum_{i=1}^m \alpha_i(t,s)\langle e_{t,s},e_i\rangle\right)^2 \geq \frac{\beta_0^2}{m}.$$

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Plugging (4.34) into (4.32) and noting that $1/\|s - t\|^m$ is integrable on $J \times J'$, we conclude that $\int \int_{D_0} A(t,s)\,dt\,ds$ is finite and super-exponentially small.

Next we show that $\int \int_{D_i} A(t,s)\,dt\,ds$ is super-exponentially small for $i = m+1, \ldots, k$. It follows from (4.27) that $\int \int_{D_i} A(t,s)\,dt\,ds$ is bounded by

$$
\int \int_{D_i} dt\,ds \int_u^\infty dx \int_0^\infty dw_i\, p_{X(t),\nabla X_{|J}(t),X_i(s),\nabla X_{|J'}(s)}(x,0,w_i,0)
$$
$$
\times\, \mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)| \,|\, X(t) = x, \nabla X_{|J}(t) = 0, X_i(s) = w_i, \nabla X_{|J'}(s) = 0\}.
$$
(4.35)

We can write

$$
p_{X(t),X_i(s)}(x, w_i | X_i(t) = 0) = \frac{1}{2\pi\sigma_1(t)\sigma_2(t,s)(1 - \rho^2(t,s))^{1/2}}
$$
$$
\times \exp\left\{ -\frac{1}{2(1 - \rho^2(t,s))}\left(\frac{x^2}{\sigma_1^2(t)} + \frac{w_i^2}{\sigma_2^2(t,s)} - \frac{2\rho(t,s)xw_i}{\sigma_1(t)\sigma_2(t,s)} \right) \right\},
$$

where

$$
\sigma_1^2(t) = \mathrm{Var}(X(t)|X_i(t) = 0), \quad \rho(t,s) = \frac{\mathbb{E}\{X(t)X_i(s)|X_i(t) = 0\}}{\sigma_1(t)\sigma_2(t,s)},
$$
$$
\sigma_2^2(t,s) = \mathrm{Var}(X_i(s)|X_i(t) = 0) = \frac{\det\mathrm{Cov}(X_i(s), X_i(t))}{\lambda_{ii}},
$$

and $\rho^2(t,s) < 1$ due to (**H3**). Therefore,

$$
p_{X(t),\nabla X_{|J}(t),X_i(s),\nabla X_{|J'}(s)}(x,0,w_i,0)
$$
$$
= p_{\nabla X_{|J'}(s),X_1(t),\ldots,X_{i-1}(t),X_{i+1}(t),\ldots,X_k(t)}(0|X(t) = x, X_i(s) = w_i, X_i(t) = 0)
$$
$$
\times\, p_{X(t),X_i(s)}(x, w_i | X_i(t) = 0)p_{X_i(t)}(0)
$$
$$
\leq C_3 \exp\left\{ -\frac{1}{2(1 - \rho^2(t,s))}\left(\frac{x^2}{\sigma_1^2(t)} + \frac{w_i^2}{\sigma_2^2(t,s)} - \frac{2\rho(t,s)xw_i}{\sigma_1(t)\sigma_2(t,s)} \right) \right\}
$$
$$
\times\, (\det\mathrm{Cov}(X(t), \nabla X_{|J}(t), X_i(s), \nabla X_{|J'}(s)))^{-1/2}
$$
(4.36)

for some positive constant C_3. Also, by similar argument in the proof of Theorem 4.7, there exist positive constants C_4, C_5, C_6, C_7, N_2 and N_3 such that

$$
\det\mathrm{Cov}(\nabla X_{|J}(t), X_i(s), \nabla X_{|J'}(s)) \geq C_4\|s - t\|^{2(m+1)},
$$
(4.37)

$$
C_5\|s - t\|^2 \leq \sigma_2^2(t,s) \leq C_6\|s - t\|^2,
$$
(4.38)

and

$$
\mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)| \,|\, X(t) = x, \nabla X_{|J}(t) = 0, X_i(s) = w_i, \nabla X_{|J'}(s) = 0\}
$$
$$
= \mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)| \,|\, X(t) = x, \nabla X_{|J}(t) = 0,
$$
$$
\langle \nabla X_i(t), e_{t,s}\rangle = w_i/\|s - t\| + o(1), \nabla X_{|J'}(s) = 0\}
$$
$$
\leq C_7(x^{N_2} + (w_i/\|s - t\|)^{N_3} + 1).
$$
(4.39)

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Combining (4.35) with (4.36), (4.37) and (4.39), and making change of variable $w = w_i/\|s - t\|$, we obtain that for some positive constant C_8,

$$
\begin{aligned}
\int &\int_{D_i} A(t, s)\, dtds \\
&\le C_8 \int \int_{D_i} dtds \|s - t\|^{-m-1} \int_u^\infty dx \int_0^\infty dw_i (x^{N_2} + (w_i/\|s - t\|)^{N_3} + 1) \\
&\quad \times \exp\left\{ -\frac{1}{2(1 - \rho^2(t, s))}\left(\frac{x^2}{\sigma_1^2(t)} + \frac{w_i^2}{\sigma_2^2(t, s)} - \frac{2\rho(t, s)xw_i}{\sigma_1(t)\sigma_2(t, s)}\right)\right\} \\
&= C_8 \int \int_{D_i} dtds \|s - t\|^{-m} \int_u^\infty dx \int_0^\infty dw (x^{N_2} + w^{N_3} + 1) \\
&\quad \times \exp\left\{ -\frac{1}{2(1 - \rho^2(t, s))}\left(\frac{x^2}{\sigma_1^2(t)} + \frac{w^2}{\widetilde{\sigma}_2^2(t, s)} - \frac{2\rho(t, s)xw}{\sigma_1(t)\widetilde{\sigma}_2(t, s)}\right)\right\},
\end{aligned}
\tag{4.40}
$$

where $\widetilde{\sigma}_2(t, s) = \sigma_2(t, s)/\|s - t\|$ is bounded by (4.38). Applying Taylor's formula (4.22) to $X_i(s)$ and noting that $\mathbb{E}\{X(t)\langle \nabla X_i(t), e_{t,s}\rangle\} = -\alpha_i(t, s)$, we obtain

$$
\begin{aligned}
\rho(t, s) &= \frac{1}{\sigma_1(t)\sigma_2(t, s)}\left(\mathbb{E}\{X(t)X_i(s)\} - \frac{1}{\lambda_{ii}}\mathbb{E}\{X(t)X_i(t)\}\mathbb{E}\{X_i(s)X_i(t)\}\right) \\
&= \frac{\|s - t\|}{\sigma_1(t)\sigma_2(t, s)}\left(-\alpha_i(t, s) + \|s - t\|^\eta \mathbb{E}\{X(t)Y_{t,s}^i\}\right. \\
&\qquad \left. - \frac{\|s - t\|^\eta}{\lambda_{ii}}\mathbb{E}\{X(t)X_i(t)\}\mathbb{E}\{X_i(t)Y_{t,s}^i\}\right).
\end{aligned}
$$

By (4.38) and the fact that $(t, s) \in D_i$ implies $\alpha_i(t, s) \ge \beta_i > 0$ for $i = m + 1, \ldots, k$, we conclude that $\rho(t, s) \le -\delta_0$ for some $\delta_0 > 0$ uniformly for $t \in J$, $s \in J'$ with $\|s - t\|$ sufficiently small. Then applying Lemma 4.10 to (4.40) yields that $\int \int_{D_i} A(t, s)\, dtds$ is super-exponentially small.

It is similar to prove that $\int \int_{D_i} A(t, s)\, dtds$ is super-exponentially small for $i = k + 1, \ldots, k + k' - m$. In fact, in such case, $\int \int_{D_i} A(t, s)\, dtds$ is bounded by

$$
\begin{aligned}
\int &\int_{D_i} dtds \int_u^\infty dx \int_0^\infty dz_i\, p_{X(t), \nabla X_{|J}(t), X_i(t), \nabla X_{|J'}(s)}(x, 0, z_i, 0) \\
&\times \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)| \,|\, X(t) = x, \nabla X_{|J}(t) = 0, X_i(t) = z_i, \nabla X_{|J'}(s) = 0\}.
\end{aligned}
$$

We can follow the proof in the previous stage by exchanging the positions of $X_i(s)$ and $X_i(t)$ and replacing w_i with z_i. The details are omitted since the procedure is very similar.

If $k = 0$, then $m = 0$ and $\sigma(J') = \{1, \ldots, k'\}$. Since J becomes a singleton, we may let $J = \{t_0\}$. By the Kac-Rice metatheorem, $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded by

$$\int_{J'} ds \int_u^\infty dx \int_u^\infty dy \int_0^\infty dz_1 \cdots \int_0^\infty dz_{k'} \, p_{t_0,s}(x, y, z_1, \ldots, z_{k'}, 0)$$
$$\times \, \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)| \, | \, X(t_0) = x, X(s) = y, X_1(t_0) = z_1, \ldots, X_{k'}(t_0) = z_{k'}, \nabla X_{|J'}(s) = 0\}$$
$$:= \int_{J'} \widetilde{A}(t_0, s) \, ds,$$

where $p_{t_0,s}(x, y, z_1, \ldots, z_{k'}, 0)$ is the density of $(X(t_0), X(s), X_1(t_0), \ldots, X_{k'}(t_0), \nabla X_{|J'}(s))$ evaluated at $(x, y, z_1, \ldots, z_{k'}, 0)$. Similarly, J' could be covered by $\cup_{i=1}^{k'} \widetilde{D}_i$ with $\widetilde{D}_i = \{s \in J' : \alpha_i(t_0, s) \leq -\widetilde{\beta}_i\}$ for some positive constants $\widetilde{\beta}_i$, $1 \leq i \leq k'$. On the other hand,

$$\int_{\widetilde{D}_i} \widetilde{A}(t_0, s) \, ds \leq \int_{\widetilde{D}_i} ds \int_u^\infty dx \int_0^\infty dz_i \, p_{X(t_0), X_i(t_0), \nabla X_{|J'}(s)}(x, z_i, 0)$$
$$\times \, \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)| \, | \, X(t_0) = x, X_i(t_0) = z_i, \nabla X_{|J'}(s) = 0\}.$$

By similar discussions, we obtain that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small and hence complete the proof. $\square$

5 Further Remarks and Examples

Remark 5.1 [The case when T contains the origin] We now show that Theorem 4.7 and Theorem 4.8 still hold when T contains the origin. In such case, (**H3**) is actually not satisfied since $X(0) = 0$ is degenerate. However, we may construct a small open cube T_0 containing 0 such that $\sup_{t \in T_0} \sigma_t^2$ is sufficiently small, then according to the Borell-TIS inequality, $\mathbb{P}\{\sup_{t \in T_0} X(t) \geq u\}$ is super-exponentially small. Let $\widehat{T} = T \backslash T_0$, then

$$\mathbb{P}\Big\{\sup_{t \in \widehat{T}} X(t) \geq u\Big\} \leq \mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} \leq \mathbb{P}\Big\{\sup_{t \in \widehat{T}} X(t) \geq u\Big\} + \mathbb{P}\Big\{\sup_{t \in T_0} X(t) \geq u\Big\}. \quad (5.1)$$

To estimate $\mathbb{P}\{\sup_{t \in \widehat{T}} X(t) \geq u\}$, similarly to the rectangle T, we decompose $\widehat{T}$ into several faces by lower dimensions such that $\widehat{T} = \cup_{k=0}^N \partial_k \widehat{T} = \cup_{k=0}^N \cup_{L \in \partial_k \widehat{T}} L$. Then we can get the bounds similar to (4.3) with T replaced with $\widehat{T}$ and J replaced with L. Following the proof of Theorem 4.7 yields

$$\mathbb{P}\Big\{\sup_{t \in \widehat{T}} X(t) \geq u\Big\} = \sum_{k=0}^N \sum_{L \in \partial_k \widehat{T}} \mathbb{E}\{M_u(L)\} + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}).$$

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Due to the fact that $\sup_{t \in T_0} \sigma_t^2$ is sufficiently small, $\mathbb{E}\{M_u(L)\}$ are super-exponentially small for all faces L such that $L \subset \partial_k \bar{T}_0$ with $0 \le k \le N-1$ (note that $\bar{T}_0$ is a compact rectangle). The same reason yields that for $1 \le k \le N$, $L \in \partial_k \widehat{T}$, $J \in \partial_k T$ such that $L \subset J$, the difference between $\mathbb{E}\{M_u(L)\}$ and $\mathbb{E}\{M_u(J)\}$ is super-exponentially small. Hence we obtain

$$\mathbb{P}\Big\{\sup_{t \in \widehat{T}} X(t) \ge u\Big\} = \sum_{\{t\} \in \partial_0 T} \Psi\Big(\frac{u}{\sigma_t}\Big) + \sum_{k=1}^{N} \sum_{J \in \partial_k T} \frac{1}{(2\pi)^{(k+1)/2} |\Lambda_J|^{1/2}}$$
$$\times \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\Big(\frac{u}{\theta_t}\Big) e^{-u^2/(2\theta_t^2)} \, dt + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}). \tag{5.2}$$

Here, by convention, if $\theta_t = 0$, we regard $e^{-u^2/(2\theta_t^2)}$ as 0. Combining (5.1) with (5.2), we conclude that Theorem 4.7 still holds when T contains the origin. The argument for Theorem 4.8 is similar.

Remark 5.2 Based on the proofs of Theorems 4.7 and 4.8, one may expect that the approximation (1.1) holds for a much wider class of smooth Gaussian fields (not necessarily with stationary increments). Meanwhile, the argument for the parameter set could go far beyond the rectangle case. These further developments will be included in Cheng (2013).

Example 5.3 [Refinements of Theorem 4.7] Let X be a Gaussian random field as in Theorem 4.7. Suppose that $\nu(t_0) = \sigma_T^2$ for some $t_0 \in J \in \partial_k T$ ($k \ge 0$) and $\nu(t) < \sigma_T^2$ for all $t \ne t_0$.

(i). If $k = 0$, then, due to (4.14), $\sup_{t \in T \setminus \{t_0\}} \theta_t^2 \le \sigma_T^2 - \varepsilon_0$ for some $\varepsilon_0 > 0$. This implies that $\mathbb{E}\{M_u(J')\}$ is super-exponentially small for every face J' other than $\{t_0\}$. Therefore,

$$\mathbb{P}\Big\{\sup_{t \in T} X(t) \ge u\Big\} = \Psi\Big(\frac{u}{\sigma_T}\Big) + o(e^{-u^2/(2\sigma_T^2) + \alpha u^2}), \quad \text{as } u \to \infty. \tag{5.3}$$

For example, let Y be a stationary Gaussian field with covariance $\rho(t) = e^{-\|t\|^2}$ and define $X(t) = Y(t) - Y(0)$, then X is a smooth Gaussian field with stationary increments satisfying conditions (**H1**)-(**H3**). Let $T = [0,1]^N$, then we can apply (5.3) to approximate the excursion probability of X with $t_0 = (1, \ldots, 1)$.

(ii). If $k \ge 1$, then similarly, $\mathbb{E}\{M_u(J')\}$ is super-exponentially small for every face $J' \ne J$. It follows from Theorem 4.7 that

$$\mathbb{P}\Big\{\sup_{t \in T} X(t) \ge u\Big\} = \frac{u^{k-1}}{(2\pi)^{(k+1)/2} |\Lambda_J|^{1/2}} \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^{2k-1}} e^{-u^2/(2\theta_t^2)} \, dt(1 + o(1)).$$

Let $\tau(t) = \theta_t^2$, then $\forall i \in \sigma(J)$, $\tau_i(t_0) = 0$, since t_0 is a local maximum point of τ restricted on J. Assume additionally that the Hessian matrix

$$\Theta_J(t_0) := (\tau_{ij}(t_0))_{i,j \in \sigma(J)} \tag{5.4}$$

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is negative definite, then the Hessian matrix of $1/(2\theta_t^2)$ at t_0 restricted on J,

$$\widetilde{\Theta}_J(t_0) = -\frac{1}{2\tau^2(t_0)}(\tau_{ij}(t_0))_{i,j\in\sigma(J)} = -\frac{1}{2\sigma_T^4}\Theta_J(t_0),$$

is positive definite. Let $g(t) = |\Lambda_J - \Lambda_J(t)|/\theta_t^{2k-1}$ and $h(t) = 1/(2\theta_t^2)$, applying Lemma 6.3 with T replaced with J gives us that as $u \to \infty$,

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \frac{u^{k-1}|\Lambda_J - \Lambda_J(t_0)|}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_{t_0}^{2k-1}}\frac{(2\pi)^{k/2}}{u^k|\widetilde{\Theta}_J(t_0)|^{1/2}}e^{-u^2/(2\theta_{t_0}^2)}(1+o(1))$$

$$= \frac{2^{k/2}|\Lambda_J - \Lambda_J(t_0)|}{|\Lambda_J|^{1/2}| - \Theta_J(t_0)|^{1/2}}\Psi\Big(\frac{u}{\sigma_T}\Big)(1+o(1)). \tag{5.5}$$

Example 5.3 [Continued: the cosine field] We consider the *cosine random field* on $\mathbb{R}^2$:

$$Z(t) = \frac{1}{\sqrt{2}}\sum_{i=1}^{2}(\xi_i\cos t_i + \xi_i'\sin t_i), \quad t = (t_1, t_2) \in \mathbb{R}^2,$$

where ξ_1, ξ_1', ξ_2, ξ_2' are independent, standard Gaussian variables. Z is a well-known centered, unit-variance and smooth stationary Gaussian field [cf. Adler and Taylor (2007, p.382)]. Note that Z is periodic and $Z(t) = -Z_{11}(t) - Z_{22}(t)$. To avoid such degeneracy, let $X(t) = \xi_0 + Z(t) - Z(0)$, where $t \in T \subset [0, 2\pi)^2$ and ξ_0 is a standard Gaussian variable independent of Z. Then X is a centered and smooth Gaussian field with stationary increments. The variance and covariance of X are given respectively by

$$\nu(t) = \sigma_t^2 = 3 - \cos t_1 - \cos t_2,$$

$$C(t, s) = 2 + \frac{1}{2}\sum_{i=1}^{2}[\cos(t_i - s_i) - \cos t_i - \cos s_i]. \tag{5.6}$$

Therefore, X satisfies conditions (**H1**), (**H2**) and (**H3**) [though $X_{12}(t) \equiv 0$, it can be shown that this does not affect the validity of Theorems 4.7 and 4.8]. Taking the partial derivatives of C gives us that

$$\mathbb{E}\{X(t)\nabla X(t)\} = \frac{1}{2}(\sin t_1, \sin t_2)^T, \quad \Lambda = \mathrm{Cov}(\nabla X(t)) = \frac{1}{2}I_2,$$

$$\Lambda - \Lambda(t) = -\mathbb{E}\{X(t)\nabla^2 X(t)\} = \frac{1}{2}[I_2 - \mathrm{diag}(\cos t_1, \cos t_2)], \tag{5.7}$$

where I_2 is the 2×2 unit matrix and diag denotes the diagonal matrix.

(i). Let $T = [0, \pi/2]^2$. Then by (5.6), ν attains its maximum 3 only at the corner $(\pi/2, \pi/2)$, where both partial derivatives of ν are positive. Applying the result (i) in Example 5.3, we obtain $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\} = \Psi(u/\sqrt{3})(1+o(e^{-\alpha u^2}))$.

(ii). Let $T = [0, 3\pi/2] \times [0, \pi/2]$. Then ν attains its maximum 4 only at the boundary point $t^* = (\pi, \pi/2)$, where $\nu_2(t^*) > 0$ so that the condition (4.14) is satisfied. In this case, $t^* \in J = (0, 3\pi/2) \times \{\pi/2\}$. By (5.7), we obtain $\Lambda_J = \frac{1}{2}$ and $\Lambda_J - \Lambda_J(t^*) = \frac{1}{2}(1 - \cos t_1^*) = 1$. On the other hand, for $t \in J$, by Lemma 6.1 and (5.7),

$$\tau(t) = \theta_t^2 = \mathrm{Var}(X(t)|X_1(t)) = 3 - \cos t_1 - \cos t_2 - \frac{1}{2}\sin^2 t_1, \tag{5.8}$$

therefore $\Theta_J(t^*) = \tau_{11}(t^*) = -2$. Plugging these into (5.5) with $k = 1$ gives us that $\mathbb{P}\{\sup_{t \in T} X(t) \geq u\} = \sqrt{2}\Psi(u/2)(1 + o(1))$.

(iii). Let $T = [0, 3\pi/2]^2$. Then ν attains its maximum 5 only at the interior point $t^* = (\pi, \pi)$. In this case, $t^* \in J = (0, 3\pi/2)^2$. By (5.7), we obtain $\Lambda_J = \frac{1}{2}I_2$ and $\Lambda_J - \Lambda_J(t^*) = I_2$. On the other hand, for $t \in J$, by Lemma 6.1 and (5.7),

$$\tau(t) = \theta_t^2 = \mathrm{Var}(X(t)|X_1(t), X_2(t)) = 3 - \cos t_1 - \cos t_2 - \frac{1}{2}\sin^2 t_1 - \frac{1}{2}\sin^2 t_2, \tag{5.9}$$

therefore $\Theta_J(t^*) = (\tau_{ij}(t^*))_{i,j=1,2} = -2I_2$. Plugging these into (5.5) with $k = 2$ gives us that $\mathbb{P}\{\sup_{t \in T} X(t) \geq u\} = 2\Psi(u/\sqrt{5})(1 + o(1))$.

Example 5.4 [Refinements of Theorem 4.8] Let X be a Gaussian field as in Theorem 4.8. Suppose $t_0 \in J \in \partial_k T$ is the only point in T such that $\nu(t_0) = \sigma_T^2$. Assume $\sigma(J) = \{1, \ldots, k\}$, all elements in $\varepsilon(J)$ are 1, $\nu_{k'}(t_0) = 0$ for all $k+1 \leq k' \leq N$. Then by Theorem 4.8,

$$\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} = \mathbb{E}\{M_u^E(J)\} + \sum_{k'=k+1}^{N} \sum_{J' \in \partial_{k'} T, \bar{J}' \cap \bar{J} \neq \emptyset} \mathbb{E}\{M_u^E(J')\} + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}). \tag{5.10}$$

Lemma 4.3 indicates $\mathbb{E}\{M_u^E(J)\} = (-1)^k \mathbb{E}\{\sum_{i=0}^{k}(-1)^i \mu_i(J)\}(1 + o(e^{-\alpha x^2}))$, therefore

$$\mathbb{E}\{M_u^E(J)\} = (-1)^k \int_J p_{\nabla X_{|J}(t)}(0)dt\, \mathbb{E}\{\det \nabla^2 X_{|J}(t)\mathbb{1}_{\{(X_{k+1}(t), \ldots, X_N(t)) \in \mathbb{R}_+^{N-k}\}}$$
$$\times \mathbb{1}_{\{X(t) \geq u\}}|\nabla X_{|J}(t) = 0\}(1 + o(e^{-\alpha x^2}))$$
$$= \int_u^{\infty} dx \int_J dt\, \frac{(-1)^k e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t} \mathbb{E}\{\det \nabla^2 X_{|J}(t)\mathbb{1}_{\{(X_{k+1}(t), \ldots, X_N(t)) \in \mathbb{R}_+^{N-k}\}}$$
$$|X(t) = x, \nabla X_{|J}(t) = 0\}(1 + o(e^{-\alpha u^2}))$$
$$:= \int_u^{\infty} A_J(x)dx(1 + o(e^{-\alpha u^2})), \tag{5.11}$$

and similarly,

$$\mathbb{E}\{M_u^E(J')\} = \int_u^{\infty} dx \int_{J'} dt\, \frac{(-1)^{k'} e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k'+1)/2}|\Lambda_{J'}|^{1/2}\theta_t} \mathbb{E}\{\det \nabla^2 X_{|J'}(t)$$
$$\times \mathbb{1}_{\{(X_{J_1'}(t), \ldots, X_{J_{N-k'}'}(t)) \in \mathbb{R}_+^{N-k'}\}}|X(t) = x, \nabla X_{|J'}(t) = 0\}(1 + o(e^{-\alpha u^2})).$$

(i). First we consider the case $k \geq 1$. We shall follow the notation $\tau(t)$, $\Theta_J(t)$ and $\widetilde{\Theta}_J(t)$ in Example 5.3. Let $h(t) = 1/(2\theta_t^2)$ and

$$g_x(t) = \frac{(-1)^k}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t}\mathbb{E}\{\det\nabla^2 X_{|J}(t)\mathbb{1}_{\{(X_{k+1}(t),\ldots,X_N(t))\in\mathbb{R}_+^{N-k}\}}$$

$$|X(t) = x, \nabla X_{|J}(t) = 0\}.$$

Note that $\sup_{t\in T}|g_x(t)| = o(x^{N_1})$ for some $N_1 > 0$ as $x \to \infty$, which implies that the growth of $g_x(t)$ can be dominated by the exponential decay $e^{-x^2 h(t)}$, hence both Lemma 6.3 and 6.4 are still applicable. Applying Lemma 6.3 with T replaced by J and u replaced by x^2, we obtain that as $x \to \infty$,

$$A_J(x) = \frac{(2\pi)^{k/2}}{x^k(\det\widetilde{\Theta}_J(t_0))^{1/2}}g_x(t_0)e^{-x^2/(2\sigma_T^2)}(1 + o(1)). \tag{5.12}$$

On the other hand, it follows from (3.17) that

$$g_x(t) = \frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t}\int\cdots\int_{\mathbb{R}_+^{N-k}}dy_{k+1}\cdots dy_N$$

$$\times \frac{|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k}H_k\left(\frac{x}{\gamma_t} + \gamma_t C_{k+1}(t)y_{k+1} + \cdots + \gamma_t C_N(t)y_N\right)$$

$$\times p_{X_{k+1}(t),\ldots,X_N(t)}(y_{k+1},\ldots,y_N|X(t) = x, \nabla X_{|J}(t) = 0).$$

Note that $X(t_0)$ and $\nabla X(t_0)$ are independent, and $C_j(t_0) = 0$ for all $1 \leq j \leq N$. Therefore,

$$g_x(t_0) = \frac{|\Lambda_J - \Lambda_J(t_0)|}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\sigma_T^{k+1}}H_k\left(\frac{x}{\sigma_T}\right)$$

$$\times \mathbb{P}\{(X_{k+1}(t_0),\ldots,X_N(t_0)) \in \mathbb{R}_+^{N-k}|\nabla X_{|J}(t_0) = 0\}.$$

Plugging this and (5.12) into (5.11), we obtain

$$\mathbb{E}\{M_u^E(J)\} = \frac{2^{k/2}|\Lambda_J - \Lambda_J(t_0)|}{|\Lambda_J|^{1/2}| - \Theta_J(t_0)|^{1/2}}\Psi\left(\frac{u}{\sigma_T}\right)$$

$$\times \mathbb{P}\{(X_{k+1}(t_0),\ldots,X_N(t_0)) \in \mathbb{R}_+^{N-k}|\nabla X_{|J}(t_0) = 0\}(1 + o(1)). \tag{5.13}$$

For $J' \in \partial_{k'}T$ with $\bar{J}' \cap \bar{J} \neq \emptyset$, similarly, applying Lemma 6.4 with T replaced by J', we obtain that

$$\mathbb{E}\{M_u^E(J')\} = \frac{2^{k'/2}|\Lambda_{J'} - \Lambda_{J'}(t_0)|}{|\Lambda_{J'}|^{1/2}| - \Theta_{J'}(t_0)|^{1/2}}\Psi\left(\frac{u}{\sigma_T}\right)\mathbb{P}\{Z_{J'}(t_0) \in \mathbb{R}_-^{k'-k}\}$$

$$\times \mathbb{P}\{(X_{J_1'}(t_0),\ldots,X_{J_{N-k'}'}(t_0)) \in \mathbb{R}_+^{N-k'}|\nabla X_{|J'}(t_0) = 0\}(1 + o(1)), \tag{5.14}$$

where $Z_{J'}(t_0)$ is a centered $(k' - k)$-dimensional Gaussian vector with covariance matrix $-(\tau_{ij})_{i,j \in \sigma(J') \backslash \sigma(J)}$. Plugging (5.13) and (5.14) into (5.10), we obtain the asymptotic result.

(ii). $k = 0$, say $J = \{t_0\}$. Note that $X(t_0)$ and $\nabla X(t_0)$ are independent, therefore

$$\mathbb{E}\{M_u^E(J)\} = \Psi\left(\frac{u}{\sigma_T}\right)\mathbb{P}\{\nabla X(t_0) \in \mathbb{R}_+^N\}. \tag{5.15}$$

For $J' \in \partial_{k'}T$ with $\bar{J}' \cap \bar{J} \neq \emptyset$, then $\mathbb{E}\{M_u^E(J')\}$ is given by (5.14) with $k = 0$. Plugging (5.15) and (5.14) into (5.10), we obtain the asymptotic formula for the excursion probability.

Example 5.4 [Continued: the cosine field] We consider the Gaussian field X defined in the continued part of Example 5.3.

(i). Let $T = [0, \pi]^2$. Then ν attains its maximum 5 only at the corner $t^* = (\pi, \pi)$, where $\nabla\nu(t^*) = 0$ so that the condition (4.14) is not satisfied. Instead, we will use the result (ii) in Example 5.4 with $J = \{t^*\}$ and $k = 0$. Let $J' = (0, \pi) \times \{\pi\}$, $J'' = \{\pi\} \times (0, \pi)$. Combining the results in the continued part of Example 5.3 with (5.15) and (5.14), and noting that $\Lambda = \frac{1}{2}I_2$ implies $X_1(t)$ and $X_2(t)$ are independent for all t, we obtain

$$\mathbb{E}\{M_u^E(J)\} = \frac{1}{4}\Psi(u/\sqrt{5}), \quad \mathbb{E}\{M_u^E(\partial_2 T)\} = \frac{1}{2}\Psi(u/\sqrt{5})(1 + o(1)),$$

$$\mathbb{E}\{M_u^E(J')\} = \mathbb{E}\{M_u^E(J'')\} = \frac{\sqrt{2}}{4}\Psi(u/\sqrt{5})(1 + o(1)).$$

Summing these up, we have $\mathbb{P}\{\sup_{t \in T} X(t) \geq u\} = [(3 + 2\sqrt{2})/4]\Psi(u/\sqrt{5})(1 + o(1))$.

(ii). Let $T = [0, 3\pi/2] \times [0, \pi]$. Then ν attains its maximum 5 only at the boundary point $t^* = (\pi, \pi)$, where $\nu_2(t^*) = 0$. Applying the result (i) in Example 5.4 with $J = (0, 3\pi/2) \times \{\pi\}$ and $k = 1$, we obtain

$$\mathbb{E}\{M_u^E(J)\} = \frac{\sqrt{2}}{2}\Psi(u/\sqrt{5}), \quad \mathbb{E}\{M_u^E(\partial_2 T)\} = \Psi(u/\sqrt{5})(1 + o(1)),$$

which implies $\mathbb{P}\{\sup_{t \in T} X(t) \geq u\} = [(2 + \sqrt{2})/2]\Psi(u/\sqrt{5})(1 + o(1))$.

Remark 5.5 Note that we have only provided the first-order approximations for the examples in this section. However, as shown in the theory of approximations of integrals [see e.g. Wong (2001)], the integrals in (4.15) and (4.26) can be expanded with more terms provided the covariance function of the Gaussian field is smooth enough. Hence for the examples above, higher-order approximation is available. Since the procedure is similar and the computation is tedious, we omit such argument here.

6   Appendix

This Appendix contains proofs of Lemmas 4.1, 4.9, 4.10, and some other auxiliary facts.

Proof of Lemma 4.1 By the definition of $M_u^E(J)$, it is clear that

$$\left\{ \sup_{t \in T} X(t) \geq u \right\} \supset \bigcup_{k=0}^{N} \bigcup_{J \in \partial_k T} \{M_u^E(J) \geq 1\} \text{ a.s.}$$

Suppose $\sup_{t \in T} X(t) \geq u$, since $X(t) \in C^2(\mathbb{R}^N)$ a.s., there exists $t_0 \in T$ such that $X(t_0) = \sup_{t \in T} X(t)$. Without loss of generality, assume $t_0 \in J \in \partial_k T$. Note that t_0 is a local maximum restricted on J, thus $\nabla X_{|J}(t_0) = 0$ and $\nabla^2 X_{|J}(t_0)$ is non-positive definite. Due to (**H**1) and (**H**3$'$), we apply Lemma 11.2.11 in Adler and Taylor (2007) to obtain that almost surely, $\det(\nabla^2 X_{|J}(t_0)) \neq 0$ and hence $\text{index}(\nabla^2 X_{|J}(t_0)) = k$. If $\varepsilon_j^* X_j(t_0) < 0$ for some $j \notin \sigma(J)$, then we can find $t_1 \in T$ such that $X(t_1) > X(t_0)$, which contradicts $X(t_0) = \sup_{t \in T} X(t)$. Hence $\varepsilon_j^* X_j(t_0) \geq 0$ for all $j \notin \sigma(J)$. These indicate $M_u^E(J) \geq 1$, therefore

$$\left\{ \sup_{t \in T} X(t) \geq u \right\} \subset \bigcup_{k=0}^{N} \bigcup_{J \in \partial_k T} \{M_u^E(J) \geq 1\} \text{ a.s.,}$$

completing the proof. $\qquad\square$

Proof of Lemma 4.9 Let $M_{N \times N}$ be the set of all $N \times N$ matrices. Define a mapping $\phi : \mathbb{R}^N \times M_{N \times N} \to \mathbb{R}$ by $(\xi, A) \mapsto \langle \xi, A\xi \rangle$, then ϕ is continuous. Since $\Lambda - \Lambda(t)$ is positive definite, $\phi(e, \Lambda - \Lambda(t)) > 0$ for each $t \in T$ and $e \in \mathbb{S}^{N-1}$. On the other hand, $\{(e, \Lambda - \Lambda(t)) : t \in T, e \in \mathbb{S}^{N-1}\}$ is a compact subset of $\mathbb{R}^N \times M_{N \times N}$ and ϕ is continuous, completing the proof. $\qquad\square$

Proof of Lemma 4.10 We only prove the first case, since the second case follows from the first one. By elementary computation on the joint density of $\xi_1(t)$ and $\xi_2(s)$, we obtain

$$\sup_{t \in T_1, s \in T_2} \mathbb{E}\{(1 + |\xi_1(t)|^{N_1} + |\xi_2(s)|^{N_2}) \mathbb{1}_{\{\xi_1(t) \geq u, \xi_2(s) < 0\}}\}$$

$$\leq \frac{1}{2\pi \underline{\sigma}_1 \underline{\sigma}_2 (1 - \overline{\rho}^2)^{1/2}} \int_u^\infty \exp\left\{ -\frac{x_1^2}{2\overline{\sigma}_1^2} \right\} dx_1$$

$$\int_{-\infty}^0 (1 + |x_1|^{N_1} + |x_2|^{N_2}) \exp\left\{ -\frac{1}{2\overline{\sigma}_2^2(1 - \underline{\rho}^2)} \left(x_2 - \frac{\sigma_2 \underline{\rho} x_1}{\overline{\sigma}_1} \right)^2 \right\} dx_2$$

$$= o\left(\exp\left\{ -\frac{u^2}{2\overline{\sigma}_1^2} - \frac{\sigma_2^2 \underline{\rho}^2 u^2}{2\overline{\sigma}_2^2(1 - \underline{\rho}^2)\overline{\sigma}_1^2} + \varepsilon u^2 \right\} \right),$$

as $u \to \infty$, for any $\varepsilon > 0$. $\qquad\square$

The following lemma is well-known and is quoted here for reader's convenience.

Lemma 6.1 *Let Y and Z be two Gaussian random vectors of dimension p and q, respectively. Then $Y|Z = z$ is still a p-dimensional Gaussian random vector having the following mean and covariance:*

$$\mathbb{E}\{Y|Z = z\} = \mathbb{E}Y + \mathbb{E}\{(Y - \mathbb{E}Y)(Z - \mathbb{E}Z)^T\}[\mathrm{Cov}(Z)]^{-1}(z - \mathbb{E}Z),$$

$$\mathrm{Cov}(Y|Z = z) = \mathrm{Cov}(Y) - \mathbb{E}\{(Y - \mathbb{E}Y)(Z - \mathbb{E}Z)^T\}[\mathrm{Cov}(Z)]^{-1}\mathbb{E}\{(Z - \mathbb{E}Z)(Y - \mathbb{E}Y)^T\}.$$

In particular, if $p = q = 1$ and $\mathbb{E}Y = \mathbb{E}Z = 0$, then

$$\mathbb{E}\{Y|Z = z\} = \frac{z\mathbb{E}(YZ)}{\mathrm{Var}(Z)}, \quad \mathrm{Var}(Y|Z = z) = \mathrm{Var}(Y) - \frac{(\mathbb{E}(YZ))^2}{\mathrm{Var}(Z)}.$$

Using similar argument in the proof of Lemma 4.9, we can obtain the following result.

Lemma 6.2 *Let $\{A(t) = (a_{ij}(t))_{1 \leq i,j \leq N} : t \in T\}$ be a family of positive definite matrices such that all elements $a_{ij}(\cdot)$ are continuous. Denote by $\underline{x}$ and $\overline{x}$ the infimum and supremum of the eigenvalues of $A(t)$ over $t \in T$ respectively, then $0 < \underline{x} \leq \overline{x} < \infty$.*

The following two formulas state the results on the Laplace method approximation. Lemma 6.3 can be found in many books on the approximations of integrals, here we refer to Wong (2001). Lemma 6.4 can be derived by following similar argument in the proof of Laplace method for the case of boundary point in Wong (2001) [see Cheng (2013)].

Lemma 6.3 *[Laplace method for interior point] Let t_0 be an interior point of T. Suppose the following conditions hold: (i) $g(t) \in C(T)$ and $g(t_0) \neq 0$; (ii) $h(t) \in C^2(T)$ and attains its unique minimum at t_0; and (iii) $\nabla^2 h(t_0)$ is positive definite. Then as $u \to \infty$,*

$$\int_T g(t)e^{-uh(t)}dt = \frac{(2\pi)^{N/2}}{u^{N/2}(\det\nabla^2 h(t_0))^{1/2}}g(t_0)e^{-uh(t_0)}(1 + o(1)).$$

Lemma 6.4 *[Laplace method for boundary point] Let $t_0 \in J \in \partial_k T$ with $0 \leq k \leq N - 1$. Suppose that conditions (i), (ii) and (iii) in Lemma 6.3 hold, and additionally $\nabla h(t_0) = 0$. Then as $u \to \infty$,*

$$\int_T g(t)e^{-uh(t)}dt = \frac{(2\pi)^{N/2}\mathbb{P}\{Z_J(t_0) \in (-E(J))\}}{u^{N/2}(\det\nabla^2 h(t_0))^{1/2}}g(t_0)e^{-uh(t_0)}(1 + o(1)),$$

where $Z_J(t_0)$ is a centered $(N - k)$-dimensional Gaussian vector with covariance matrix $(h_{ij}(t_0))_{J_1 \leq i,j \leq J_{N-k}}$, $-E(J) = \{x \in \mathbb{R}^N : -x \in E(J)\}$, and the definitions of $J_1, \ldots, J_{N-k}$ and $E(J)$ are in (3.4).

References

[1] R. J. Adler (1981), *The Geometry of Random Fields.* Wiley, New York.

[2] R. J. Adler (2000), On excursion sets, tube formulas and maxima of random fields. *Ann. Appl. Probab.* **10**, 1-74.

[3] R. J. Adler and J. E. Taylor (2007), *Random Fields and Geometry.* Springer, New York.

[4] R. J. Adler and J. E. Taylor (2011), *Topological complexity of smooth random functions.* Lecture Notes in Mathematics, 2019. École d'Été de Probabilités de Saint-Flour. Springer, Heidelberg.

[5] R. J. Adler, J. E. Taylor and K. J. Worsley (2012), *Applications of Random Fields and Geometry: Foundations and Case Studies. In preparation.*

[6] J.-M. Azaïs, J.-M. Bardet and M. Wschebor (2002), On the tails of the distribution of the maximum of a smooth stationary Gaussian process. *ESAIM Probab. Statist.* **6**, 177–184.

[7] J.-M. Azaïs and C. Delmas (2002), Asymptotic expansions for the distribution of the maximum of Gaussian random fields. *Extremes.* **5**, 181–212.

[8] J.-M. Azaïs and M. Wschebor (2005), On the distribution of the maximum of a Gaussian field with d parameters. *Ann. Appl. Probab.* **15**, 254–278.

[9] Azaïs, J.-M. and Wschebor, M. (2008), A general expression for the distribution of the maximum of a Gaussian field and the approximation of the tail. *Stoch. Process. Appl.* **118** (2008), 1190–1218.

[10] J.-M. Azaïs and M. Wschebor (2009), *Level Sets and Extrema of Random Processes and Fields.* John Wiley & Sons, Hoboken, NJ.

[11] K. Bartz, S. C. Kou and R. J. Adler (2011), Estimating thresholding levels for random fields via Euler characteristics. *Preprint.*

[12] D. Cheng (2013), *Excursion Probability of Gaussian and Asymptotically Gaussian Random Fields.* Ph.D. Thesis, Michigan State University. *In preparation.*

[13] J. Liu (2012), Tail approximations of integrals of Gaussian random fields. *Ann. Probab.* **40**, 1069–1104.

[14] G. Matheron (1973), The intrinsic random functions and their applications. *Adv. Appl. Probab.* **5**, 439–468.

[15] V. I. Piterbarg (1996a), *Asymptotic Methods in the Theory of Gaussian Processes and Fields.* Translations of Mathematical Monographs, Vol. 148, American Mathematical Society, Providence, RI.

[16] V. I. Piterbarg (1996b), Rice's method for large excursions of Gaussian random fields. Technical Report NO. 478, Center for Stochastic Processes, Univ. North Carolina.

[17] J. Potthoff (2010), Sample properties of random fields III: differentiability. *Commun. Stoch. Anal.* **4**, 335-353.

[18] M. L. Stein (1999), *Interpolation of Spatial Data: Some Theory for Kriging.* Springer, New York.

[19] M. L. Stein (2012), On a class of space-time intrinsic random functions. *Bernoulli*, to appear.

[20] J. Sun (1993), Tail Probabilities of the Maxima of Gaussian Random Fields. *Ann. Probab.* **21**, 34–71.

[21] A. Takemura and S. Kurki (2002), On the equivalence of the tube and Euler characteristic methods for the distribution of the maximum of Gaussian fields over piecewise smooth domains. *Ann. Appl. Probab.* **12**, 768-796.

[22] J. E. Taylor and R. J. Adler (2003), Euler characteristics for Gaussian fields on manifolds. *Ann. Probab.* **31**, 533–563.

[23] J. E. Taylor, A. Takemura, and R. J. Adler (2005), Validity of the expected Euler characteristic heuristic. *Ann. Probab.* **33**, 1362–1396.

[24] R. Wong (2001), *Asymptotic Approximations of Integrals.* SIAM, Philadelphia, PA.

[25] Y. Xiao (2009), Sample path properties of anisotropic Gaussian random fields. In: *A Minicourse on Stochastic Partial Differential Equations*, (D. Khoshnevisan and F. Rassoul-Agha, editors), *Lecture Notes in Math.* **1962**, pp. 145–212. Springer, New York.

[26] Y. Xue and Y. Xiao (2011), Fractal and smoothness properties of space-time Gaussian models. *Frontiers Math. China* **6**, 1217–1246.

[27] A. M. Yaglom (1957), Some classes of random fields in n-dimensional space, related to stationary random processes. *Th. Probab. Appl.* **2**, 273–320.

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